225 PROBLEM SET 5, SPRING 2007

DUE TUESDAY, APRIL 3

(1) (a) (5 points) Let A be the matrix that describes the projection of the plane onto a line l throughthe origin. Explain from a geometric point of view what you expect A^2 to be and why.

(b) (5 points) Write down the formula for the projection of x onto the line spanned by $\begin{pmatrix} a \\ b \end{pmatrix}$.

(c) (5 points) Write down the matrix A describing the projection onto the line spanned by $\begin{pmatrix} a \\ b \end{pmatrix}$.

(d) (5 points) Compute A^2, and compare the result with your answer to (a).

(e) (5 points) From a geometric point of view, what do you think is $\det(A)$ and why?

(f) (5 points) Compute $\det(A)$ to check your answer to the previous question.

(g) (5 points) Is A an orthogonal matrix?

(h) (5 points) What is the orthogonal projection of $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ to the line spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

(i) (5 points) What is the orthogonal projection of $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ to the line spanned by $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$?

Date: March 27, 2007.

(j) (5 points) Whitout further calculation, express $\begin{pmatrix} 8 \\ 2 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, and explain why your answer works. [Hint: if you don't know what to do and want to cheat a little, you can calculate what the linear combination should be and then interpret it using the answers to the previous questions.]

(2) (a) (5 points) Let A be the matrix that describes the reflection of the plane at a line l throughthe origin. Explain from a geometric point of view what you expect A^2 to be and why.

(b) (5 points) Write down the formula for the reflection of x at the line spanned by $\begin{pmatrix} a \\ b \end{pmatrix}$.

(c) (5 points) Write down the matrix A describing the reflection at the line spanned by $\begin{pmatrix} a \\ b \end{pmatrix}$.

(d) (5 points) Compute A^2, and compare the result with your answer to (a).

(e) (5 points) Is A an orthogonal matrix?

(f) (5 points) From a geometric point of view, what do you think is $\det(A)$ and why?

(g) (5 points) Compute $\det(A)$ to check your answer to the previous question.

(h) (5 points) What is the reflection of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ at the line spanned by $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$?

(3) **Fourier series:** We are going to work with the vector space $\mathcal{L}^2([-\pi,\pi])$ of integrable complex valued functions on the interval $[-\pi,\pi]$. [In case you don't know the complex numbers,

you may take your functions to be real valued.] On this vector space, we define the following inner product:

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot\overline{g(x)}dx.$$

Or, if you prefer working over the real numbers,

$$\langle f, g\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot g(x)dx.$$

(a) (10 points) Show that $\langle -, -\rangle$ is indeed an inner product. Hint: to show this, you need to show that it satisfies:

  (i) $\langle f_1 + f_2, g\rangle = \langle f_1, g\rangle + \langle f_2, g\rangle$
  (ii) $\langle \lambda f, g\rangle = \lambda\langle f, g\rangle$, where λ is a (complex) scalar.
  (iii) $\langle f_1, g_1 + g_2\rangle = \langle f, g_1\rangle + \langle f, g_2\rangle$
  (iv) $\langle f, \lambda g\rangle = \overline{\lambda}\langle f, g\rangle$, where λ is a complex scalar. [If λ is real, then $\overline{\lambda} = \lambda$.]

(b) (optional) Show that the functions

$$f_n(x) = e^{ix}$$

with $n \in \mathbb{Z}$ form an orthonormal system. If you are working over the reals, you need to consider two kinds of functions:

$$f_n(x) = \cos(nx)$$

for $n \geq 0$ and

$$g_n(x) = \sin(nx)$$

for $n \geq 1$.

[Hint: to show this, you have to check the following conditions:

  (i) For every $n \in \mathbb{Z}$, we need $\langle f_n, f_n\rangle = 1$, (and, if you are working over the reals, the same for g_n),
  (ii) For $n \neq m \in \mathbb{Z}$, we need $\langle f_n, f_m\rangle = 0$, (and, if you are working over the reals, the same for g_n and g_m), and
  (iii) For every $n, m \in \mathbb{N}$, we need $\langle f_n, g_m\rangle = 0$ (you only need to do this part if you are working over the reals.)

(c) (optional) What is the orthogonal projection of

$$h(x) = \pi^2 - x^2$$

to $\sin(nx)$? Using integration by parts, calculate the orthogonal projection of $h(x)$ to $\cos(nx)$.
Hint: you can find the solution at

$$http://webpages.dcu.ie/\sim applebyj/ms224/BT2\_FSER.pdf$$