FIRST 347 MIDTERM, FALL 2005
DUE WEDNESDAY 9/28

You are not allowed to discuss these problems with anybody apart from me.

(1) Let X be a set, and let $R \subseteq X \times X$ be a subset of the product of X with itself. For $x, y \in X$, we write $x \sim y$ if $(x, y) \in R$ and $x \nsim y$ if the pair (x, y) is not in R. We assume that the following statements are true for all elements x, y and z of X:

(a) $x \sim x$
(b) $x \sim y \iff y \sim x$
(c) $((x \sim y) \wedge (y \sim z)) \Rightarrow x \sim z$

For every $x \in X$, we define the subset $[x] \subseteq X$ by

$$[x] := \{y \in X \mid x \sim y\}.$$

Using only (a), (b) and (c), prove:

(d) $\forall x \in X : x \in [x]$
(e) $\forall x, y \in X : (x \in [y] \iff y \in [x])$
(f) If $x \sim y$ then $[y] \subseteq [x]$.
(g) If $y \in [x]$ then the sets $[x]$ and $[y]$ are equal, i.e.,

$$\forall x, y \in X : (y \in [x] \Rightarrow [x] = [y]).$$

(h) If $x \nsim y$ then the sets $[x]$ and $[y]$ are disjoint, i.e.,

$$\forall x, y \in X : (x \nsim y \Rightarrow [x] \cap [y] = \{\}).$$

Once you have proved a statement (d), (e), (f), (g) or (h), you are allowed to use it in order to prove the others. I think that it makes sense to prove them in the order stated, but you don't have to. If you cannot prove a statement, you are still allowed to use it for proving the ones below it. Be careful to avoid circular arguments!

Date: October 7, 2005.

Solution :

(d) Start by noting that by (a) $\forall x \in X: \; x \sim x$. Hence by definition of [x], and the fact that $x \sim x$ we have $x \in [x]$, which is the desired conclusion.

(e) $\forall x, y \in X : x \in [y] \iff$,by definition of [y], $y \sim x \iff$,in the light of the condition given in (b), $x \sim y \iff$,by definition of [x], $y \in [x]$; which is the desired conclusion.

(f) Suppose that $x \sim y$, and pick any arbitrary element $z \in [y]$. Now by definition of [y] we have $y \sim z$. Now $x \sim y$ and $y \sim z$ together with condition (c) yields $x \sim z$; thus by definition of [x] we have $z \in [x]$; whence $[y] \subseteq [x]$.

(g) If $y \in [x]$, then by definition of [x] we have $x \sim y$ which yields by the conclusion in (f) $[y] \subseteq [x]$. On the other hand $x \sim y$ together with condition (b) yields $y \sim x$, and again by the conclusion in (f) we have $[x] \subseteq [y]$. Now $[y] \subseteq [x]$ and $[x] \subseteq [y]$ together leads to the assertion

$$\forall x, y \in X : (y \in [x] \Rightarrow [x] = [y]) .$$

(h) Suppose the contrary that for some $x, y \in X$ we have

$$x \nsim y \; and \; [x] \cap [y] \neq \{\}.$$

In this case $\exists z \in [x] \cap [y]$, which means by definition of $[x]$, and $[y]$ we have $x \sim z$, and $y \sim z$. But then by the condition (b) we get $z \sim y$. And now in the light of condition (c) $x \sim z$, and $z \sim y$ together yield $x \sim y$, which is a contradiction to the hypothesis $x \nsim y$ we took for granted at the beginning. Hence negating our expression we conclude;
If $x \nsim y$ then the sets $[x]$ and $[y]$ are disjoint, i.e.,

$$\forall x, y \in X : (x \nsim y \Rightarrow [x] \cap [y] = \{\})$$

.

(2) Prove the following statement: If x is an integer such that x^2 is not a multiple of three, then x itself is not a multiple of 3. You will prove the converse "if x is not a multiple of 3 then x^2 is not a multiple of 3" in Problem 5, and you are allowed to use this fact for Problem 3.

Solution :

By the division algorithm for any $x \in \mathbb{Z}$ upon division by 3 we have

$$x = 3q + r \; for \; some \; q \in \mathbb{Z} \; and \; r \in \{0, 1, 2\}$$

. In this case we have

$$x^2 = 9q^2 + 6qr + r^2 \; for \; some \; q \in \mathbb{Z} \; and \; r \in \{0, 1, 2\}$$

. Since $9q^2 + 6qr = 3\underbrace{(3q^2 + 2qr)}_{\in \mathbb{Z}}$, and hence divisible by 3, the hypothesis x^2 is not divisible by 3, i.e. not a multiple of 3, leads to the conclusion that r^2 is not divisible by 3. Since for $r \in \{0, 1, 2\}$ we have $r^2 \in \{0, 1, 4\}$ respectively, this is possible only if $r^2 \neq 0 \iff r \neq 0$ which means that

$$x = 3q + r \; for \; some \; q \in \mathbb{Z} \; and \; r \in \{1, 2\}.$$

which translates to saying that x is not a multiple of 3. Thus, to sum up; If x is an integer such that x^2 is not a multiple of three, then x itself is not a multiple of 3.

Alternative solution :

We prove the contrapositive "if x is a multiple of 3, the so is x^2". Let x be a multiple of three, in other words, assume that there exists a $k \in \mathbb{Z}$ with $x = 3k$. Then $x^2 = 9k^2 = 3(3k^2)$ is also a multiple of 3.

(3) You can find the proof that $\sqrt{2}$ is not a rational number in the book as Example 8.13. It is an application of the proof by contradiction method. Modify the proof in the book in order to prove that $\sqrt{3}$ is not a rational number.

Solution :

Suppose the contrary that $\sqrt{3} \in \mathbb{Q}$, i.e. a rational number. Then we must have

$$\sqrt{3} = \frac{a}{b} \; for \; some \; a, b \in \mathbb{Z} \; with \; b \neq 0.$$

Furthermore, if a and b have any common factor we can make the simplification and get rid of all possible common factors. Hence, without loss of generality we may assume that $gcd(a, b) = 1$, i.e. a and b have no common factor. Then squaring both sides of the above equality yields

$$3 = \frac{a^2}{b^2}$$

which leads to the equality $a^2 = 3b^2$. But then by the contrapositive form of the assertion given in problem (5)-(e) since a^2 is divisible by 3 a itself must be divisible by 3. Thus $a = 3k$ for some $k \in \mathbb{Z}$. Plugging this into the equality $a^2 = 3b^2$ yields $9k^2 = 3b^2 \implies b^2 = 3k^2$. Then by again the contrapositive form of the assertion in problem (5)-(e) as b^2 is divisible by 3 b itself must be divisible by 3. So $b = 3l$ for some $l \in \mathbb{Z}$. But now 3 divides both a and b, which is a contradiction to the foremost assumption that a and b have no common divisors. Now by this contradiction negating our statement we conclude

$$\sqrt{3} \neq \frac{a}{b} \quad \forall a, b \in \mathbb{Z},$$

which simply means that $\sqrt{3} \notin \mathbb{Q}$, i.e. $\sqrt{3}$ is not a rational number.

(4) In the arena at the Colosseum there are 100 ravenous lions and a Christian. The most ravenous lion is offered the opportunity to eat the Christian. If he declines, then the Christian is set free, but if he eats the Christian, then that lion is miraculously converted into a Christian, and the spectacle continues. Assuming that the rules have been announced to the lions, determine what happens. (I recommend to try a proof that uses the principle of induction).

<u>**Solution :**</u>

Consider the scenario of the question with n lions instead of 100. We are going to prove by induction that the following statement $P(n)$ is true for all $n \in \mathbb{N}$.

$P(n)$: If n is odd, the lion will eat the Christian. If n is even, he won't eat the Christian.

After we proved that we may conclude that the answer to the question is "if there are 100 lions the Christian will be set free."

Basis step:
If there is only one lion, he knows that once he is converted into a Christian there is no lion left to eat him. He has nothing to fear and therefore eats the Christian.

Induction step:
Assume that we have already proved $P(n)$ and now want to prove $P(n+1)$. Let there be $n+1$ lions in the arena. The first lion has to decide whether he dares to eat the Christian. He knows that if he decides to eat him, he will be turned into a Christian and left with n ravenous lions. Since we (and the first lion) know $P(n)$ to be true, this means that if n is odd (and thus $n+1$ even) he expects the next lion to eat him, whereas if n is even (and thus $n+1$ is odd) he expects that the next lion won't eat him. Thus if $n+1$ is even, the first of the $n+1$ lions is afraid to be eaten and won't eat the Christian, while if $n+1$ is odd he has nothing to fear and will happily eat the Christian.

(5) Let $\mathbb{Z}$ be the set of all integers, and consider the set

$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \text{ is a multiple of } 3\}.$$

(a) Show that R satisfies conditions (a), (b) and (c) of Problem 1. Once you have proved this, you know that the statements (d), (e), (f), (g) and (h) of Problem 1 are also true for R, and you are allowed to use them.

Solution :

Condition (a): $\forall x \in \mathbb{Z}$ we have $x - x = 0 = 3 \cdot 0$. Hence by the definition of set R we have $(x, x) \in R \ \ \forall x \in \mathbb{Z}$, which means $x \sim x \ \ \forall x \in \mathbb{Z}$, as desired.

Condition (b): $\forall x, y \in \mathbb{Z} \ \ x \sim y \Longrightarrow (x, y) \in R \Longrightarrow x - y$

is a multiple of $3 \Longrightarrow x - y = 3k$ for some $k \in \mathbb{Z} \Longrightarrow -(x - y) = -3k = 3 \cdot (-k) \Longrightarrow y - x = 3 \cdot (-k)$ where

$(-k) \in \mathbb{Z} \Longrightarrow y - x$ is a multiple of $3 \Longrightarrow (y, x) \in R \Longrightarrow y \sim x$. Thus

$$x \sim y \Longrightarrow y \sim x.$$

Starting with $y \sim x$ and exchanging the places of y and x in the reasoning above we get $x \sim y$ similary. Hence

$$y \sim x \Longrightarrow x \sim y.$$

And uniting the both expressions we have

$$x \sim y \Longleftrightarrow y \sim x.$$

which is the desired conclusion.

Condition (c): $\forall x, y, z \in \mathbb{Z} \;\; x \sim y$, and $y \sim z \Longrightarrow (x, y) \in$

R and $(y, z) \in R \Longrightarrow x - y$ is a multiple of 3, and $y - z$ is a multiple of $3 \Longrightarrow x - y = 3k$, and $y - z = 3l$ for some $k, l \in \mathbb{Z} \Longrightarrow x - z = x - y + y - z = 3k + 3l = 3(k + l)$ where $(k + l) \in \mathbb{Z}$, which means $x - z$ is a multiple of $3 \Longrightarrow (x, z) \in R \Longrightarrow x \sim z$, as desired.

(b) Any integer n can be written in the form

$$n = 3k + r,$$

whith $k \in \mathbb{Z}$ and $r \in \{0, 1, 2\}$, where r is the remainder of n when divided by 3 (you don't need to prove this fact). Prove that for every $n \in \mathbb{Z}$,

$$[n] = [0] \quad \text{or} \quad [n] = [1] \quad \text{or} \quad [n] = [2].$$

Solution :

For any $n \in \mathbb{Z}$ we have

$n = 3k + r$ for some $k \in \mathbb{Z}$ and $r \in \{0, 1, 2\}$

$\Longrightarrow n - r = 3k$, a multiple of 3, which implies $(n, r) \in R \Longrightarrow n \sim r \Longrightarrow$ by condition (b) of $\sim$; $r \sim n \Longrightarrow n \in [r] \Longrightarrow$, by conclusion in Problem (1)-(g), $[n] = [r]$.Hence summing up our reasoning

$\forall n \in \mathbb{Z} \quad [n] = [r]$ for some $r \in \{0, 1, 2\}$

, which translates to simply for every $n \in \mathbb{Z}$,

$$[n] = [0] \quad \text{or} \quad [n] = [1] \quad \text{or} \quad [n] = [2],$$

as desired.

(c) Write down the first four positive elements of each of the sets [0], [1] and [2].

Solution :

By definition

$$[0] = \{n \in \mathbb{Z} : n - 0 = n \quad \text{is a multiple of } 3\}$$

Hence

$$[0] = \{ \quad \ldots, -6, -3, 0, 3, 6, 9, 12, \ldots \quad \}.$$

Thus, the first four positive elements of [0] are $3, 6, 9, 12$. Similary

$$[1] = \{n \in \mathbb{Z} : n - 1 \ = \ 3k \text{ for some } k \in \mathbb{Z}\} \Longrightarrow$$

$$[1] = \{n \in \mathbb{Z} : n \ = \ 3k + 1 \text{ for some } k \in \mathbb{Z}\} = \{3k + 1 : k \in \mathbb{Z}\} =$$

$$\{\ldots, 3 \cdot (-2) + 1, 3 \cdot (-1) + 1, 3 \cdot (0) + 1, 3 \cdot 1 + 1, 3 \cdot 2 + 1, 3 \cdot 3 + 1, \ldots\}$$

$$= \{ \quad \ldots, -5, -2, 1, 4, 7, 10, \ldots, \quad \}.$$

Thus, the first four positive elements of [1] are $1, 4, 7, 10$.In exactly the same fashion

$$[2] = \{3k + 2 \quad : \quad k \in \mathbb{Z}\} \quad \Longrightarrow \quad [2] =$$

$$\{\ldots, 3 \cdot (-2) + 2, 3 \cdot (-1) + 2, 3 \cdot (0) + 2, 3 \cdot 1 + 2, 3 \cdot 2 + 2, 3 \cdot 3 + 2, \ldots\}$$

$$= \{ \quad \ldots, -4, -1, 2, 5, 8, 11, \ldots, \quad \}.$$

Therefore, the first four positive elements of [2] are $2, 5, 8, 11$.

(d) Prove: if $x_1 \sim x_2$ and $y_1 \sim y_2$, then

$$x_1 + y_1 \sim x_2 + y_2 \quad \text{and} \quad x_1 \cdot y_1 \sim x_2 \cdot y_2.$$

Solution :

Suppose $x_1 \sim x_2$, and $y_1 \sim y_2$ then $(x_1, x_2) \in R$, and $(y_1, y_2) \in R \Longrightarrow x_1 - x_2 = 3k$, and $y_1 - y_2 = 3l$ for some $k, l \in \mathbb{Z}$
$\Longrightarrow (x_1 + y_1) - (x_2 + y_2) = (x_1 - x_2) + (y_1 - y_2) = 3k + 3l = 3 \cdot \underbrace{(k + l)}_{\in \mathbb{Z}} \Longrightarrow (x_1 + y_1) - (x_2 + y_2)$ is a multiple of 3 $\Longrightarrow (x_1 + y_1, x_2 + y_2) \in R \Longrightarrow (x_1 + y_1) \sim (x_2 + y_2)$, which is the first result asked to prove.

Again the same hypotheses $\Longrightarrow x_1 \cdot y_1 - x_2 \cdot y_2 = x_1 \cdot (y_1 - y_2) + y_2 \cdot (x_1 - x_2) = x_1 \cdot 3l + y_2 \cdot 3k = 3 \cdot \underbrace{(x_1 \cdot l + y_2 \cdot k)}_{\in \mathbb{Z}} \Longrightarrow$ $x_1 \cdot y_1 - x_2 \cdot y_2$ is a multiple of 3 $\Longrightarrow (x_1 \cdot y_1, x_2 \cdot y_2) \in R \Longrightarrow$ $x_1 \cdot y_1 \sim x_2 \cdot y_2$., which is our second result asked to prove.

(e) Now prove that if 3 does not divide x then 3 does not divide x^2.

<u>**Solution :**</u>

If 3 does not divide x then $x \nsim 3 \Longrightarrow$, by the conclusion in problem (1)-(h), $[x] \cap [0] = \{\}$.But then our conclusion in part (b) gives us either $[x] = [1]$, or $[x] = [2]$, which means either $x \sim 1$, or $x \sim 2$, which implies by part (d) either $x^2 = x \cdot x \sim 1 \cdot 1 = 1$, or $x^2 = x \cdot x = 2 \cdot 2 = 4 \Longrightarrow$ either $x^2 - 1$ is divisible by 3, or $x^2 - 4$ is divisible by 3. Since $x^2 - 1 = x^2 - 4 + 3$, and $x^2 - 4 = x^2 - 1 + (-3)$ in which cases 3 and -3 both being divisible by 3 we conclude that $x^2 - 1$ is divisible by 3 if and only if $x^2 - 4$ is divisible by 3. But then we have the result
If x is not divisible by 3, then $[x^2] = [1]$, which translates to x^2 is not divisible by 3.
Hence, if 3 does not divide x then 3 does not divide x^2.