

FIRST 347 MIDTERM, FALL 2005
DUE WEDNESDAY 9/28

You are not allowed to discuss these problems with anybody apart from me.

- (1) Let X be a set, and let $R \subseteq X \times X$ be a subset of the product of X with itself. For $x, y \in X$, we write $x \sim y$ if $(x, y) \in R$ and $x \not\sim y$ if the pair (x, y) is not in R . We assume that the following statements are true for all elements x, y and z of X :
- (a) $x \sim x$
 - (b) $x \sim y \iff y \sim x$
 - (c) $((x \sim y) \wedge (y \sim z)) \Rightarrow x \sim z$

For every $x \in X$, we define the subset $[x] \subseteq X$ by

$$[x] := \{y \in X \mid x \sim y\}.$$

Using only (a), (b) and (c), prove:

- (d) $\forall x \in X : x \in [x]$
- (e) $\forall x, y \in X : (x \in [y] \iff y \in [x])$
- (f) If $x \sim y$ then $[y] \subseteq [x]$.
- (g) If $y \in [x]$ then the sets $[x]$ and $[y]$ are equal, i.e.,
$$\forall x, y \in X : (y \in [x] \Rightarrow [x] = [y]).$$
- (h) If $x \not\sim y$ then the sets $[x]$ and $[y]$ are disjoint, i.e.,
$$\forall x, y \in X : (x \not\sim y \Rightarrow [x] \cap [y] = \{\}).$$

Once you have proved a statement (d), (e), (f), (g) or (h), you are allowed to use it in order to prove the others. I think that it makes sense to prove them in the order stated, but you don't have to. If you cannot prove a statement, you are still allowed to use it for proving the ones below it. Be careful to avoid circular arguments!

- (2) Prove the following statement: If x is an integer such that x^2 is not a multiple of three, then x itself is not a multiple of 3. You will prove the converse “if x is not a multiple of 3 then x^2 is not a multiple of 3” in Problem 5, and you are allowed to use this fact for Problem 3.
- (3) You can find the proof that $\sqrt{2}$ is not a rational number in the book as Example 8.13. It is an application of the proof by contradiction method. Modify the proof in the book in order to prove that $\sqrt{3}$ is not a rational number.
- (4) In the arena at the Colosseum there are 100 ravenous lions and a Christian. The most ravenous lion is offered the opportunity to eat the Christian. If he declines, then the Christian is set free, but if he eats the Christian, then that lion is miraculously converted into a Christian, and the spectacle continues. Assuming that the rules have been announced to the lions, determine what happens. (I recommend to try a proof that uses the principle of induction).
- (5) Let \mathbb{Z} be the set of all integers, and consider the set

$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \text{ is a multiple of } 3\}.$$

- (a) Show that R satisfies conditions (a), (b) and (c) of Problem 1. Once you have proved this, you know that the statements (d), (e), (f), (g) and (h) of Problem 1 are also true for R , and you are allowed to use them.
- (b) Any integer n can be written in the form

$$n = 3k + r,$$

whith $k \in \mathbb{Z}$ and $r \in \{0, 1, 2\}$, where r is the remainder of n when divided by 3 (you don't need to prove this fact). Prove that for every $n \in \mathbb{Z}$,

$$[n] = [0] \quad \text{or} \quad [n] = [1] \quad \text{or} \quad [n] = [2].$$

- (c) Write down the first four positive elements of each of the sets $[0]$, $[1]$ and $[2]$.
- (d) Prove: if $x_1 \sim x_2$ and $y_1 \sim y_2$, then

$$x_1 + y_1 \sim x_2 + y_2 \quad \text{and} \quad x_1 \cdot y_1 \sim x_2 \cdot y_2.$$

- (e) Now prove that if 3 does not divide x then 3 does not divide x^2 .