347 PROBLEM SET 4, FALL 2005

(1) Negate the following statements: (X, Y and Z are sets, P and Q are properties.)
  (a) $\forall x \in X : \forall y \in Y \mid P(x,y) : \exists z \in Z : Q(y,z)$
  (b) Every student in Illinois has a friend who hates math.
  (c) All the heaters in Altgeld are not working properly.
  (d) $\exists y \in Y : \forall x \in X \mid Q(x) : P(x,y)$.
  (e) There exists a key which can open all blue doors in the house.

(2) (Bonus) Let L and P be sets, and let $R \subseteq L \times P$ be a subset of the product of L with P. We call the elements of L "lines" and the elements of P "points". We say that a point p lies on the line l if $(l,p) \in R$ and that p does not lie on l if (l,p) is not an element of R. We further say that the two lines l and l' intersect in the point q if q lies on l and q lies on l'.
  (a) Write down a formula (with quantifiers like "$\forall$" or "$\exists$", etc.) for the following statement (called "parallel axiom"): *For every line l it is true that for every point p which is not on l, there exists exactly one line l' through p such that l and l' do not intersect.*
  (b) Write down the negation of the parallel axiom as formula and in plain English. Convince yourself that this is really the opposite, compare it with your friends' results.

(3) Using induction, prove the following statement for all $n \in \mathbb{N}$, and all $q \in \mathbb{R}$ with $q \neq 1$:

$$q^0 + q^1 + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}.$$

This expression is called the *geometric sum equation.* It is proved in the book as Corollary 3.14, but you should give a direct proof of it using induction.

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