PROBLEM SET 6 – REVISED VERSION, 347, FALL 2005
DUE FRIDAY 10/14

(1) (16 points) Let $f : X \to Y$ and $g : Y \to Z$ be maps. Prove:
  (a) If $g \circ f$ is injective, then f is also injective.
  (b) If $g \circ f$ is surjective, then g is also surjective.

(2) (4 points) Write down a bijective map from $\mathbb{N}$ to $\mathbb{Z}$ (no proof required).

(3) (12 points)
  (a) Let $f : X \to Y$ be a map. Define a relation $\sim$ on X by

$$x_1 \sim x_2 \overset{\text{def}}{\iff} f(x_1) = f(x_2).$$

  Show that $\sim$ is an equivalence relation, i.e. that it satisfies conditions 1(a)–1(c) from the midterm.

  (b) (Quotient of a set by an equivalence relation) Let X be a set and let $\sim$ be an equivalence relation on X. Consider the set of all equivalence classes of elements of X:

$$\{[x] | x \in X\}.$$

  We denote this set "$X/\sim$" and call it *the quotient of X by $\sim$*. This is a set whose elements are sets. But that should not upset you! Just think of them as elements. Here is a warm-up question: How many elements does the set

$$\mathbb{Z}/\sim$$

  have, where $\sim$ is as in the example of question 5 of the midterm? Write down a complete list of its elements. How many elements does each element of this set have?

  (c) Let $(\mathbb{Z}, \sim)$ as above, and consider the map

$$\begin{aligned} f : \mathbb{Z} &\to \mathbb{Z}/\sim \\ z &\mapsto [z]. \end{aligned}$$

  This map is obviously surjective.
  What is $f(17)$? What is $f(31)$? What is $f(99)$? (By this I mean: *which of the elements of the list from part (b) of this question are they?*)

  (d) Show that in our example

$$f : \mathbb{Z} \to \mathbb{Z}/\sim$$

  is not injective.

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(4) (10 points) Let $\sim$ be an equivalence relation on X. We call $\sim$ *trivial* if the following holds:

$$x \sim y \Longleftrightarrow x = y.$$

Prove: The map

$$\begin{aligned} f : X &\rightarrow X/\sim \\ x &\mapsto [x]. \end{aligned}$$

is injective if and only if $\sim$ is trivial.

Remark: Again, this map is obviously surjective, so you actually just proved that it is bijective if and only if f is trivial.