PROBLEM SET 8, 347, FALL 2005
DUE FRIDAY 11/4

(1) (15 points) There are 10 pirates on a priate ship in the middle of the ocean, and they have 100 gold coins to divide among themselves. Here is the method they will use.

The most ferocious pirate proposes a division, and then they all vote on it. If at least half of them support the division, the division passes, otherwise, the proposer is tossed overboard and the second most ferocious pirate gets to propose a division, and so on.

Every pirate tries to get as many coins for himself as possible, but as it happens, pirates are so ornery, that if they are personally indifferent about a proposed division, they will vote against it, hoping to see someone tossed overboard.

Assuming the pirates know all this and every pirate also knows everybody's place in the hierarchy, determine what happens.

Write up a clean induction proof! You might want to look my comments on your answer to the lions question (first midterm).

(2) **Conjugacy classes of the symmetric group:** Let $\mathfrak{S}_n$ be the set of all permutations of the set $\{1, \dots, n\}$, and let

$$\begin{aligned} \circ \colon \mathfrak{S}_n \times \mathfrak{S}_n &\to \mathfrak{S}_n \\ (f, g) &\mapsto f \circ g \end{aligned}$$

be the composition of permutations.

We define a relation $\sim$ on $\mathfrak{S}_n$ as follows:

$$f_1 \sim f_2 \overset{\text{def}}{\iff} \quad \text{there exists a } g \in \mathfrak{S}_n \text{ such that } g^{-1} f_1 g = f_2.$$

If $f_1 \sim f_2$, we say "f_1 *is conjugate to* f_2".

(a) (3 points) Prove that $\sim$ is an equivalence relation. We call the equivalence class $[f]$ of a permutation f the "*conjugacy class*" of f.
(b) (6 points) Find all conjugacy classes of $\mathfrak{S}_3$. Write them down using cycle notation.
(c) (8 points) Now do the same thing for $\mathfrak{S}_4$. Make a guess about what is going on, and try to explain it. (No formal proof required)

($\mathfrak{S}_3$ has 6 elements, and $\mathfrak{S}_4$ has 24 elements.)

(3) (8 points)
(a) (2 points) Compute $12^{10000000003} \mod 13$.
(b) (6 points) Find the multiplicative inverse of 7 in $\mathbb{Z}/129\mathbb{Z}$.

Date: October 28, 2005.