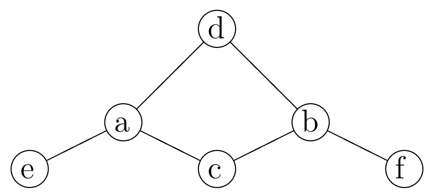
PROBLEM SET 9, 347, FALL 2005
DUE FRIDAY 11/18

(1) **(Final practice)** Let X be a set, and let $R \subseteq X \times X$ be a subset. We write $x \prec y$ for $(x, y) \in R$. We call R a (partial) order relation, if it satisfies the following three properties:

(a) $x \prec x$ (R is *reflexive*)
(b) $(x \prec y) \ \wedge \ (y \prec x) \ \Rightarrow x = y$ (R is *antisymmetric*)
(c) $(x \prec y) \ \wedge \ (y \prec z) \ \Rightarrow x \prec z$ (R is *transitive*)

Such a pair $(X, \prec)$, where $\prec$ is an order relation on X, is often called "*poset*" for "*partially ordered set*". Here is a way to visualize posets: One draws vertices and edges like for graphs, but in such a way that $x \prec y$ if and only if x is connected to y by an ascending path. For example this picture



describes the set $X = \{a, b, c, d, e, f\}$ with the relation

$$a \prec a, b \prec b, c \prec c, d \prec d, e \prec e, f \prec f,$$

$$e \prec a, e \prec d, a \prec d, c \prec a, c \prec b, c \prec d, b \prec d, f \prec b, f \prec d$$

Note how we have $e \prec d$ because of the path e, a, d, we don't need to draw an extra edge to visualize this relation. Note also how a is higher up in the picture then f, but there is no ascending path from f to a, so that $f \prec a$ does not hold.

(d) **(3 points)** Let Y be any set and let $\mathcal{P}(Y)$ be the power set of a set Y. Show that $(\mathcal{P}(Y), \subseteq)$ is a poset.

(e) **(3 points)** For $Y = \{1\}$, $Y = \{1, 2\}$ and $Y = \{1, 2, 3\}$ draw the poset of part (d).

(f) Let $(X, \prec)$ be a poset, and let $x \in X$. We define $\mathcal{I}(x)$ to be the set

$$\mathcal{I}(x) := \{y \in X \mid y \prec x\}.$$

For the set X of our example above, what is $\mathcal{I}(b)$? What is $\mathcal{I}(d)$?

Now let $(X, \prec)$ again be an arbitrary poset and let $x_1, x_2 \in X$.
Prove: **(5 points)** If $x_1 \prec x_2$, then $\mathcal{I}(x_1) \subseteq \mathcal{I}(x_2)$.
(5 points) The sets $\mathcal{I}(x_1)$ and $\mathcal{I}(x_2)$ are equal if and only if $x_1 = x_2$.

Date: November 11, 2005.

(2) **(8 points each)** Do questions 11.2, 11.15 and 11.17 from the book.

(3) **(Bonus, 15 points)** In this question you are going to show that the complex numbers are a field. Consider the set $\mathbb{C}$ of ordered pairs of real numbers

$$\mathbb{C} := \mathbb{R} \times \mathbb{R}.$$

Rather than (a, b), we will write

$$a + bi$$

for the ordered pair consisting of a and b. We define maps $+$ and $\cdot$ from $\mathbb{C} \times \mathbb{C}$ to $\mathbb{C}$ by

$$\begin{aligned} (a+bi)+(c+di) &:= (a+b)+(c+d)i \text{ and} \\ (a+bi)\cdot(c+di) &:= ac-bd+(ac+bd)i \end{aligned}$$

In particular, $i \cdot i = -1$. Note that this is very similar to what we did in Question 5 of Midterm 2: There we condidered pairs of rational numbers (a, b), and we could have written them as $a + b\sqrt{2}$ to get the multiplication from the midterm.

(a) Compute $(1 + 3i)^3$, $\frac{2+i}{3+8i}$ and $\frac{1}{2+i}$.

(b) Prove that $(\mathbb{C}, +, \cdot)$ is a field.