TAKE-HOME FINAL, FALL 2006

(1) On the very small and remote island of Fedora, all 100 inhabitants wear red hats. There are no mirrors on the island, no one ever takes off their hat, and their culture prohibits the Fedorians from ever speaking to each other about anything involving colors or hats. So everyone can see that all the others are wearing red hats, but no one knows the color of their own hat. One day a Holy Shaman, known to always tell the truth, visits the island and has a close look at its inhabitants. He tells them two things and immediately leaves. The two things are:

(a) At least one of you is wearing a red hat.

(b) Anyone who discovers for sure that their hat is red, must leave the island by midnight of the same day (s)he makes this discovery.

No Fedorian would ever leave Fedora voluntarily, but we also know that they must do whatever the Holy Shaman told them to do. What will happen after the Holy Shaman leaves?

[**Hint:** The inductive argument is over n, where n out of the 100 Fedorians on the island are wearing red hats and the other $100 - n$ are wearing different colored hats.]

(2) Go back to Problem 1 of our first take-home midterm (last years take-home final). This was the problem with the "*waters*" list. With the notation from that problem:

(a) Prove: For any two subsets S_1 and S_2 of X, we have

$$(S_1 \cup S_2)' = S_1' \cap S_2',$$

and similarly, for $T_1, T_2 \subseteq Y$, we have

$$(T_1 \cup T_2)' = T_1' \cap T_2'.$$

(b) For both statements in (a), give an example from the "*waters*" list.

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(c) The following two statements are not always true:

$$(S_1 \cap S_2)' = S_1' \cup S_2',$$
$$(T_1 \cap T_2)' = T_1' \cup T_2'.$$

Give a counterexample for each of them.

(d) Then prove: for any two subsets S_1 and S_2 of X, we have

$$(S_1 \cap S_2)' \supseteq S_1' \cup S_2',$$

and similarly for $T_1, T_2 \subseteq Y$.

(3) Consider the function

$$\begin{array}{rcl} f\colon \mathbb{R}_+ & \to & \mathbb{R}_+ \\ x & \mapsto & \sqrt{x} \end{array}$$

(i.e., $f(x) = \sqrt{x}$). Prove the following statement about this function f:

For every $x_0 \in \mathbb{R}_+$ and for every positive real number ε, there exists a positive real number δ such that for every $x \in \mathbb{R}_+$, we have:

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

[Note that the first "and" in this statement is not a logical "$\wedge$". $\mathbb{R}_+$ does not include 0.] [**Hint:** At some point in the proof, you will have to give yourself an arbitrary ε and eventually prove that something is less than ε. Note that if this works for a small epsilon, it will definitely work for a larger epsilon as well. So you can assume, without loss of generality, that epsilon is smaller than, say, $2\sqrt{x_0}$. If you do that, no complicated case analysis will be necessary. Just say in the beginning that without loss of generality you may assume epsilon to be that small and add one line at the end of your proof stating why this implies that the statement works for bigger epsilon, too.]

(4) Let $f : X \to Y$ and $g : Y \to Z$ be (well-defined) maps. Recall that the composite $g \circ f$ of f and g is the map from X to Z whose value on $x \in X$ is given by

$$(g \circ f)(x) = g(f(x)).$$

(a) Prove: If the composite $g \circ f : X \to Z$ is surjective, then g is surjective.

(b) Prove: If the composite $g \circ f$ is injective, then f is injective.