FIRST TAKE-HOME MIDTERM, FALL 2006
DUE FRIDAY 10/13

You are not allowed to discuss these problems with anybody apart from me and the TAs. You are allowed to use, without proof, everything that we did in class and everything you proved on former homework assignements.

If you do not know how to approach a problem, try to use the list from homework number 4. Writing down cleanly what you need to prove, how your proof would start and what exactly you need to show will give you partial credit. Remember that we will evaluate you on how well you write down the proof, so do not just write down a few lines of calculations, but write down formal proofs using whole sentences.

(1) **(35 points)** Do Problem (1) of last year's take-home final.

(2) **(15 points)** We consider the function (= map)

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ f(x) &:= x^2. \end{aligned}$$

Prove (formally, not with calculus-type arguments) the following statement:

For every real number x and every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for any real number y with $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.

(3) Fix a finite set X. We define the set $\mathcal{S}(X)$ as follows:

$$\mathcal{S}(X) := \{Y \mid Y \subseteq X\}.$$

(So the elements of the set $\mathcal{S}(X)$ are themselves sets, and a set Y is an element of the set $\mathcal{S}(X)$ if and only if Y is a subset of X).

(a) **(4 points):** List all the elements of $\mathcal{S}(\emptyset)$, of $\mathcal{S}(\{x\})$, of $\mathcal{S}(\{x, y\})$ and of $\mathcal{S}(\{x, y, z\})$. Make an educated guess how to fill in the blank in the following sentence:

Date: October 7, 2006.

Let n be the number of elements of X. Then the number of elements of $\mathcal{S}(X)$ equals...

(b) **(13 points)** Use the principle of induction to prove your guess from part (a).

(4) **(14 points):** Fix a natural number m, and consider the equivalence relation of the example in Problem (2) of Problem Set 5 with 3 replaced by m. Let Y be the set of all equivalence classes of this equivalence relation. I.e.,

$$Y := \{[x] \mid x \in \mathbb{Z}\} = \{[0], [1], \ldots, [m-1]\}.$$

Careful, the elements of Y are sets, which in turn contain integers. We define the operation $*$ as follows: for two classes $[x]$ and $[y]$ in Y, the class $[x] * [y]$ is given by

$$[x] * [y] := [x + y].$$

You have proved in Problem Set 5, Problem (2d) that $*$ is well defined (at least in the case that $m = 3$, but one could prove it for arbitrary m in exactly the same way).

Give complete proofs for the following statements:

(a) There exists a class $[x_0] \in Y$ such that for every class $[a] \in Y$, one has $[a] * [x_0] = [a]$.

(b) For any class $[b] \in Y$ there exists a class $[c] \in Y$ such that $[b] * [c] = [x_0]$.

(5) What is the largest real number r such that, whenever x is a real number with absolute value $|x| < r$, the inequality

$$x^2 \leq 3|x|$$

holds?

(a) **(3 points)** Find r.

(b) **(7 points)** Write down a formal proof that, with your r from part (a), for every real number x whose absolute value $|x|$ is strictly less then r the equation

$$x^2 \leq 3|x|$$

holds.

(c) **(9 points)** Prove that your r is the largest real number with this property.