FIRST IN-CLASS MIDTERM, FALL 2006
MATH 347

(1) **(? points)** Let $f\colon X \to Y$ be a map. By definition,
  (a) f is injective, if and only if

  For $x_1, x_2 \in X$,
$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

  (b) f is surjective, if and only if

  For every $y \in Y$, there exists (at least) one $x \in X$ such that $f(x) = y$.

  (c) f is bijective, if and only if

  f is injective and f is surjective.

(2) What is the negation of the following statement:
  There is a country where everybody can speak French.

  In each country, there exists at least one person who cannot speak French.

(3) Which of the following are equivalent to $A \Rightarrow B$?
  (a) $B \Rightarrow A$ NO
  (b) $(notB) \Rightarrow (notA)$ YES
  (c) $A \vee (notB)$ NO
  (d) $B \vee (notA)$ YES

  What is the negation of $A \Rightarrow B$?

$$(notA) \wedge B$$

Date: November 4, 2006.

(4) Let Z be a set, let P and Q be properties that elements of Z could have, and let

$$X = \{x \in Z \mid P(x)\}$$

and

$$Y = \{y \in Z \mid Q(y)\}.$$

(a) Express the statement $X \subseteq Y$ in terms of the properties P and Q.

$$(\forall x \in Z)\,(p(x) \Rightarrow Q(x))$$

(b) What is the set $\{z \in Z \mid P(z) \wedge Q(z)\}$ in terms of X and Y?

$$X \cap Y$$

(c) What is $Z \setminus X$ (the complement of X in Z) in terms of P and Q?

$$\{z \in Z \mid not P(z)\}$$

(d) Express the statement $(Z \setminus Y) \subseteq X$ in terms of the properties P and Q.

$$(\forall z \in Z)\,(not Q(z) \Rightarrow P(z))$$

(e) For each of the above questions, pick an example for Z, P and Q and draw the corresponding diagram.

I wanted to see Venn-diagrams of specific examples. Here are some examples:

(i) $Z = \{1, 2, 3, 4\}$, $P(x)$ is "x is even", $Q(x)$ is $x > 1$.

(ii) Z as above, $P(x)$ as above $Q(x) :\iff x \geq 3$. (then $X \cap Y = \{4\}$).

(iii) Z and $P(x)$ as above, then the complement of X in Z is the set of odd numbers in Z, i.e., $\{1, 3\}$.

(iv) Z as above, $P(x)$ as above, $Q(x) :\iff x \leq 3$. (if x is an element of Z which is not less or equal to 3, then x is 4, and 4 is even.

(5) Only one of the following three lines describes a set:

(a) $X := \{n \in \mathbb{N} \mid 2n\}$ NO
(b) $X := \{m \in \mathbb{Q} \mid (\exists n \in \mathbb{N} : 2n = m)\}$ YES
(c) $X := \forall n \in \mathbb{N} \mid 2n = m\}$ NO

(6) Which one is the set?

(b)

(7) What set is it?

The set of all even natural numbers

(8) Using the X from above, complete the following sentences:

- a is an element of X if and only if ...

  ...there exists an $n \in \mathbb{N}$ such that $2n = a$.

- m is an element of X if and only if ...

  ...there exists an $n \in \mathbb{N}$ such that $2n = m$.

- n is an element of X if and only if ...

  ...there exists a $k \in \mathbb{N}$ such that $2k = n$.

(9) For $r \in \mathbb{R}$, let

$$I(r) := \{a \in \mathbb{Q} \mid a \leq r\}.$$

(a) Let $s \in \mathbb{R}$. What is $I(s)$? What is $I(3)$?

$$I(s) := \{a \in \mathbb{Q} \mid a \leq s\}.$$
$$I(3) := \{a \in \mathbb{Q} \mid a \leq 3\}.$$

(b) Under which circumstances is r an element of $I(r)$? Give a (very short) formal reason for your answer. if and only

if r is a rational number. Reason: in order for r to be an element of $I(r)$, r needs to be a rational number and r needs to satisfy $r \leq r$. For every rational number r, we have $r \leq r$, so that part of the statement is a tautology.

(c) Let $r, s \in \mathbb{R}$. What are the sets $I(r) \cap I(s)$ and $I(r) \cup I(s)$?

$$I(r) \cap I(s) = I(\min(s,t))$$
$$I(r) \cup I(s) = I(\max(s,t).$$

(10) Consider the following two statements:

For every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that for all real numbers x and y with $|x-y| < \delta$, one has $|f(x) - f(y)| < \varepsilon$.

For every real number x and every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that for all real numbers y with $|x-y| < \delta$, one has $|f(x) - f(y)| < \varepsilon$.

(a) One of the two statements is stronger than the other (by that I mean that it implies the other one). Which one is it? Explain.

The first statement is stronger than the second one. The reason is the order of the quantifiers: in the first statement, there is one δ which has to work for all x, in the second one, δ is allowed to depend on x.

(b) How would you structure the proofs of the two statements above? Write down as many of the sentences of their proofs as you can without having to think about a specific function f. (If you need to make a clever choice for something somewhere, just say "now we pick this something = ???" and continue your proof start.)

(i) Let $\varepsilon > 0$ be arbitrary but fixed.
Pick $\delta =$???. Let $x, y \in \mathbb{R}$ be arbitrary, and assume $|x-y| < \delta$. We have to show $|f(x) - f(y)| < \varepsilon$.

(ii) Let $\varepsilon > 0$ and $x \in \mathbb{R}$ be arbitrary but fixed.
Pick $\delta =$???. Let $y \in \mathbb{R}$ be arbitrary, and assume $|x-y| < \delta$. We have to show $|f(x) - f(y)| < \varepsilon$.