SOLUTIONS FOR THE FIRST TAKE-HOME MIDTERM

MATH 347, FALL 2005

(1) The subject of Problem (1) was chosen from the field *Formal Concept Analysis*.

(a) Assume that S_1 and S_2 are both subsets of X, and that $S_1 \subseteq S_2$. We want to show that $S_2' \subseteq S_1'$, i.e., that

$$y \in S_2' \Rightarrow y \in S_1'.$$

Let y be an (arbitrary) element of S_2'. By definition of S_2', that means that

$$(I) \qquad \forall b \in S_2 : \quad b \text{ has } y.$$

We need to show that our y is an element of S_1', i.e., that

$$\forall x \in S_1 : \quad x \text{ has } y.$$

Let $x \in S_1$ be arbitrary. Since $S_1 \subseteq S_2$, we may conclude that $x \in S_2$. Applying (I) for $b = x$, it follows that x has y.

(b) Similar to (a) with T's in the roles of S'.

(c) Let S be a subset of X, and let x be an element of S. We need to prove that x is an element of S''. By definition of $S'' = (S')'$, this is equivalent to proving the following statement:

$$(II) \qquad \forall y \in S' \quad : \quad x \text{ has } y.$$

To prove this, let y be an element of S'. We need to prove that (our specific, chosen) x has (this particular) y. Since y is in S', we know

$$\forall a \in X \quad : a \text{ has } y.$$

Since our x is an element of S, it follows that y has x. Since y was chosen arbitrarily, we have proved that our x satisfies (II).
The second part of (c) is similar.

(d) Let S be a subset of X. We need to prove $S' \subseteq S'''$ and $S''' \subseteq S$. Setting $T = S'$, part (c) implies

$$S' = T \subseteq T'' = S'''.$$

To prove the other inclusion, we start with $S \subseteq S''$, which we know from (c). Applying (a) with $S_1 = S$ and $S_2 = S''$, we obtain

$$S''' = S_2' \subseteq S_1' = S'.$$

In a similar way one proves $T' = T'''$.

Date: today.

(2) *I gave a long proof with five different cases in class. Later, one student convinced me that there was a more elegant proof. Here it is:*

Let $x \in \mathbb{R}$ be arbitrary. Let ε be an arbitrary positive real number. Pick

$$\delta = -|x| + \sqrt{x^2 + \varepsilon}.$$

Note that this choice of delta satisfies

$$\delta^2 + 2|x|\delta = \varepsilon$$

Pick an arbitrary $y \in \mathbb{R}$, such that $|x - y| < \delta$. We have

$$|x^2 - y^2| = |(x - y)(x + y)| = |x - y| \cdot |x + y|$$

Now the triangle inequality gives

$$|x + y| = |y - x + 2x| \leq |y - x| + |2x| = |x - y| + 2|x|.$$

It follows that

$$|x + y| < \delta + 2|x|,$$

and thus

$$|x - y| \cdot |x + y| < \delta(\delta + 2|x|) = \varepsilon.$$

Combining this with our second equality, we obtain

$$|x^2 - y^2| < \varepsilon.$$

QED

(3) The set $\mathcal{S}(X)$ is called the *power set of* X and normally people call it $\mathcal{P}(X)$ rather than $\mathcal{S}(X)$. In order not to confuse this notation with the induction notation, I will call the statement we want to prove $Q(n)$.

The statement $Q(n)$ **says:** If X has n elements, then $\mathcal{P}(X)$ has 2^n elements.

Base step Q(0): If X has zero elements, then X is the empty set, and we have seen in part (a) that $\mathcal{P}(\emptyset) = \{\emptyset\}$ has one element. Further $1 = 2^0$.

Inducive step: Assume that $Q(n)$ was already proved for $n = k$. Let X have $k + 1$ elements. Pick one of these elements and call it a. Let $Y \subset X$. Then either $a \in Y$ or a is not an element of Y. In the second case, Y is a subset of $X \setminus \{a\}$, and in the first case,

$$Y = Z \cup \{a\},$$

where Z is a subset of $X \setminus \{a\}$. Therefore, the number of subsets of X is twice the number of subsets of $X \setminus \{a\}$. By the inductive hypothesis, the number of different subsets of $\mathcal{P}(X \setminus \{a\})$ equals 2^k. It follows that the number of elements of $\mathcal{P}(X)$ is $2 \cdot 2^k = 2^{k+1}$. We have proved $Q(k + 1)$.

(4) (a) Pick $x_0 = 0$. Let $a \in \mathbb{Z}$ be arbitrary. Then we have

$$[a] * [x_0] = [a] * [0] = [a + 0] = [a].$$

(b) Let $b \in \mathbb{Z}$ be arbitrary. Set $c := -b$. Then we have

$$[b] * [c] = [b] * [-b] = [b + (-b)] = [0].$$

(5) (a) The number is 3.

(b) Let x be a real number such that

$$|x| < 3.$$

Case 1: If $|x|$ is positive, we multiply both sides of this inequality with $|x|$ and obtain

$$x^2 < 3|x|.$$

Case 2: If $x = 0$, we have

$$x^2 = 3|x|.$$

In both cases, it follows that

$$x^2 \leq 3|x|.$$

QED

(c) Assume there was a number $s > 3$ with this property. Chose an x which is strictly greater than 3 and stricly less than s (For example, choose x to be the arithmetic mean of 3 and s.) Multiplying both sides of the inequality

$$3 < x$$

with the positive number x, we get

$$3|x| = 3x < x^2,$$

a contradiction to our assumption that s satisfies the condition of the problem. QED