SECOND TAKE-HOME MIDTERM, FALL 2006
DUE FRIDAY 11/17

(1) [**Hint:** I recommend that you start by working out the example in part (g).]

(a) **(4 points):** Write down the definitions of *injective* and *well-defined*.

Definition: An equivalence relation $\sim$ is called trivial, if

$$x_1 \sim x_2 \iff x_1 = x_2.$$

Let $f\colon X \to Y$ be a map. We define the following relation on X:

$$x_1 \sim_f x_2 \overset{def}{\iff} f(x_1) = f(x_2).$$

(b) **(3 points)** Show that $\sim_f$ is an equivalence relation.

(c) **(14 points)** Prove formally that f is injective if and only if $\sim_f$ is trivial.

Consider the map

$$\begin{aligned} p\colon X &\to X/\sim_f \\ x &\mapsto [x]. \end{aligned}$$

(d) **(2 points)** Check that p is surjective.

Recall that the image of f is defined as the set

$$\mathrm{im}(f) := \{y \in Y \mid (\exists x \in X)(f(x) = y)\}.$$

Let $i\colon \mathrm{im}(f) \to Y$ denote the inclusion map of $\mathrm{im}(f)$ in Y.

(e) **(8 points)** Prove that the map

$$\begin{aligned} g\colon X/\sim_f &\to \mathrm{im}(f) \\ [x] &\mapsto f(x) \end{aligned}$$

is well-defined.

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(f) **(12 points)** Prove that g is bijective. **(2 points)** Show that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p}\big\downarrow & & \big\uparrow{\scriptstyle i} \\ X/\sim_f & \xrightarrow{g} & \operatorname{im}(f). \end{array}$$

(In other words, convince yourself that for every $x \in X$, we have

$$i(g(p(x))) = f(x).)$$

(g) **(12 points)** **Example:** Let $X = \{1,2,3,4,5,6,7,8\}$, let $Y = \{a,b,c,d,e\}$, and let f be defined by $f(1) = a$, $f(2) = e$, $f(3) = a$, $f(4) = b$, $f(5) = e$, $f(6) = e$, $f(7) = a$, and $f(8) = c$.

(i) What is $\sim_f$?
(ii) What is $\operatorname{im}(f)$?
(iii) What is $X/\sim_f$?
(iv) What is p?
(v) What is i?
(vi) What is g?

(2) (a) **(4 points)** Negate the following statement:

For every $x_0 \in \mathbb{R}$ and for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for every $x \in \mathbb{R}$, we have

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon.$$

(b) **(14 points)** Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) := \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{else} \end{cases}$$

Prove that for this f the negation of the statement in part (a) is true. Be careful to write up your proof formally and in the correct order.

(c) **(13 points)** Let f be as in part (b). Let $x_0 = 2$. Prove that for every real number $\varepsilon > 0$ there exists a real number $\delta > 0$ such that for every $x \in \mathbb{R}$, we have

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon.$$

Be careful to write up your proof formally and in the correct order.

(3) **(14 points)** Let (a_n) be the sequence with general term

$$a_n = \frac{2n+3}{n+1}.$$

Prove that there exists a real number a such that for every positive real number ε, there exists a natural number N such that for every $n \geq N$, we have

$$|a_n - a| < \epsilon.$$

Be careful to write up your proof formally and in the correct order.