THIRD 347 MIDTERM
NOVEMBER 30, 2005

(1) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be (well-defined) maps. Complete the following sentences:

(a) (4 points) By definition, f is injective if and only if

(b) (4 points) f is not injective if and only if

(c) (2 points) How would you prove a statement of the form "*if A then B*" directly?

(d) (12 points) Prove: If f is not injective, then neither is $g \circ f$.

Date: October 7, 2006.

(2) Let X be a set, and let P and Q be properties that elements of X could have. We write $P(x)$ for "x has the property P", and $Q(x)$ for "x has the property Q". Consider the sets

$$Y := \{y \in X \mid P(y)\}$$

and

$$Z := \{x \in X \mid Q(x)\}.$$

Complete the following sentences **with statements about the properties P and Q:**

(a) (4 points) $Y \subseteq Z$ if and only if

(b) (4 points) $Y \cap Z = \emptyset$ if and only if

$$Y \cup Z = \{x \in X \mid \qquad\qquad\qquad\qquad \}$$

(c) (4 points) $Y \cup Z = X$ if and only if

(3) (2 points for every correct answer and -1 point for every false answer) For each of the following statements, indicate whether it is equivalent to $A \Rightarrow B$.

(a) **Yes** **No** "not B" holds only if "not A" holds.

(b) **Yes** **No** not (A and not B)

(c) **Yes** **No** A is sufficient for B.

(d) **Yes** **No** A is necessary for B.

(e) **Yes** **No** If B holds then A holds.

(4) **Permutations:** Consider the permutation f with two-line form

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 7 & 3 & 1 & 4 & 6 \end{pmatrix}$$

(a) (4 points) Write f in cycle form.

(b) (3 points) What is the order of f?

(c) (4 points) Compute f^{-1} in cycle or in two-line form.

(d) (6 points) Compute $f^{-1} \circ f^{-1}$ in cycle or in two-line form.

(e) (4 points) What is $f \circ f \circ f \circ f \circ f \circ f \circ f \circ f \circ f \circ f$?

(5) **Modular arithmetic:**

(a) (3 points) Compute $7^{1000000}$ modulo 5.

(b) (8 points) What is the multiplicative inverse of $[4]$ in $\mathbb{Z}/29\mathbb{Z}$?

(6) (12 points) How would you structure a proof of the following statement about a function $f\colon \mathbb{R} \to \mathbb{R}$?

For every $x_0 \in \mathbb{R}$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $x \in \mathbb{R}$ the following holds:

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Remark: At some point in this proof you would have to make a clever guess for something. Just write "???" in the place of what the guess would have to be.

(7) (12 points) Use induction, starting with $n = 0$, to prove: For every $q \in \mathbb{Q}$ with $|q| < 1$ and every $n \in \mathbb{N} \cup \{0\}$,

$$\sum_{k=0}^{n} q^k = \frac{1 - q^{n+1}}{1 - q}$$