MATH 347 – PROBLEM SET 2
DUE FRIDAY SEPTEMBER 8, 2006

(1) Let X be a set, and let P and Q be properties that elements of X could have. We write $P(x)$ for "x has property P", and similarly for Q. Consider the sets

$$Y := \{y \in X \mid P(y)\}$$

and

$$Z := \{x \in X \mid Q(x)\}.$$

Complete the following sentences with statements about the properties P and Q:

Example: $Y \subseteq Z$ if and only if

$$\forall x \in X : P(x) \Rightarrow Q(x).$$

(a) (3 points) $Y = Z$ if and only if ...
(b) (3 points) $Y \cap Z = \emptyset$ if and only if ...
(c) (6 points) Fill in the blanks

$$Y \cap Z = \{x \in X \mid \qquad\qquad\qquad\qquad\qquad \}$$

$$X \setminus Z = \{x \in X \mid \qquad\qquad\qquad\qquad\qquad \}$$

(d) (4 points) $Y \cup Z = X$ if and only if ...

(2) (4 points per answer) For each of the following statements, draw its truth table (based on the truth values for A and B, and indicate whether it is equivalent to $A \Rightarrow B$ (read "A implies B" or "if B holds then A holds"), to the negation of $A \Rightarrow B$ or to neither of the two. Then say in words what the statement says.
(a) $A \Rightarrow (\text{not } B)$
(b) $(\text{not } B) \iff (\text{not } A)$
(c) $(\text{not } B) \Rightarrow (\text{not } A)$
(d) A and $(\text{not } B)$
(e) $(\text{not } A)$ or B

Date: September 1, 2006.

(3) **(20 points)** Write truth tables, based on the possible truth values of three statements A, B and C, for the following statements. Which of them are equivalent to each other?

(a) $A \vee (B \wedge C)$
(b) $A \wedge (B \wedge C)$
(c) $(A \wedge B) \wedge C$
(d) $A \wedge (B \vee C)$
(e) $(A \vee B) \wedge C$
(f) $(A \wedge B) \vee (A \wedge C)$
(g) $(A \vee B) \wedge (A \vee C)$
(h) (not A) $\vee B$
(i) not $(A \wedge B)$

(4) Negate the following statements: (X, Y and Z are sets, P and Q are properties.)

(a) **(2 points)** Lisa is old and happy.
(b) **(4 points)** Every day in the last three years, at 3 pm, his grandfather was drinking coffee or reading the newspaper.
(c) **(4 points)** $\forall x \in X : \forall y \in Y \mid P(x, y) : \exists z \in Z : Q(y, z)$
(d) **(6 points)** I have a little sister each of whose roommates has at least one red bicycle.
(e) **(2 points)** All the heaters in Altgeld are not working properly.
(f) **(4 points)** $\exists y \in Y : \forall x \in X \mid Q(x) : P(x, y)$.
(g) **(4 points)** There exists a key which can open all blue doors in the house.
(h) **(8 points)** Let L and P be sets, and let

$$R \subseteq \{(l, p) \mid l \in L \text{ and } p \in P\}.$$

be a subset of the set of all pairs (l, p) such that l is an element of L and p is an element of P. We call the elements of L "lines" and the elements of P "points". We say that a point p lies on the line l if $(l, p) \in R$ and that p does not lie on l if (l, p) is not an element of R. We further say that the two lines l and l' intersect in the point q if q lies on l and q lies on l'.

Write down a formula (with quantifiers like "$\forall$" or "$\exists$", etc.) for the following statement (called "parallel axiom"): *For every line l it is true that for every point p which is not on l, there exists exactly one line l' through p such that l and l' do not intersect.*

Then write down the negation of the parallel axiom as formula and in plain English. Convince yourself that this is really the opposite, compare it with your friends' results.

(5) **(Word order matters, 10 points):** Using quantifiers, explain the difference between the following two statements. One of them is stronger than the other (i.e., it implies the other), explain which one it is and why.
   (a) For every guy ξ in the fraternity Ξ, there exists a girl σ in the sorority Σ such that ξ has dated σ.
   (b) There is a girl σ in the sorority Σ, such that every guy ξ in the fraternity Ξ has dated σ.