PROBLEM SET 5, FALL 2006
DUE FRIDAY 10/6

(1) **(50 points)** Let X be a set, and let $R \subseteq X \times X$ be a subset of the product of X with itself. For $x, y \in X$, we write $x \sim y$ if $(x, y) \in R$ and $x \nsim y$ if the pair (x, y) is not in R. R is called an *equivalence relation* if for any elements x, y and z of X the following statements are true:

(a) $x \sim x$ (“$\sim$ is *reflexive*”),
(b) $x \sim y \iff y \sim x$ (“$\sim$ is *symmetric*”), and
(c) $((x \sim y) \wedge (y \sim z)) \Rightarrow x \sim z$ (“$\sim$ is *transitive*”).

Let $x \in X$. We define the *equivalence class of* x to be the subset $[x] \subseteq X$ defined by

$$[x] := \{y \in X \mid x \sim y\}.$$

Before you think about the following questions, you might like to start the example of Problem (2) in order to get an idea of what is going on.
Using (a), (b) and (c), prove:

(d) $\forall x \in X : x \in [x]$
(e) $\forall x, y \in X : (x \in [y] \iff y \in [x])$
(f) If $x \sim y$ then $[y] \subseteq [x]$.
(g) If $y \in [x]$ then the sets $[x]$ and $[y]$ are equal, i.e.,

$$\forall x, y \in X : (y \in [x] \Rightarrow [x] = [y]).$$

(h) If $x \nsim y$ then the equivalence classes $[x]$ and $[y]$ are disjoint, i.e.,

$$\forall x, y \in X : (x \nsim y \Rightarrow [x] \cap [y] = \{\}).$$

Once you have proved a statement (d), (e), (f), (g) or (h), you are allowed to use it in order to prove the others. I think that it makes sense to prove them in the order stated, but you don't have to. If you cannot prove a statement, you are still allowed to use it for proving the ones below it. Be careful to avoid circular arguments!

Date: September 30, 2006.

(2) **(50 points)** Let $\mathbb{Z}$ be the set of all integers, and consider the set

$$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \text{ is a multiple of } 3\}.$$

(a) Show that R satisfies conditions (a), (b) and (c) of Problem 1. Once you have proved this, you know that the statements (d), (e), (f), (g) and (h) of Problem 1 are also true for R, and you are allowed to use them.

(b) Write down the first four positive elements of each of the equivalence classes [0], [1] and [2].

(c) You are allowed to use the fact that any integer n can be written in the form

$$n = 3k + r,$$

whith $k \in \mathbb{Z}$ and $r \in \{0, 1, 2\}$, where r is the remainder of n when divided by 3. Prove that for every $n \in \mathbb{Z}$,

$$[n] = [0] \quad \text{or} \quad [n] = [1] \quad \text{or} \quad [n] = [2].$$

(d) Prove: if $x_1 \sim x_2$ and $y_1 \sim y_2$, then

$$x_1 + y_1 \sim x_2 + y_2 \quad \text{and} \quad x_1 \cdot y_1 \sim x_2 \cdot y_2.$$

(e) Now prove that if x is an integer which is not a multiple of 3 then x^2 is also not a multiple of 3. (Suggestion: use your results from point (c)and (d)).