SOLUTIONS FOR PROBLEM SET 1
FALL 2006

Let $r_1, \ldots, r_{11}$ be eleven arbitrarily chosen real numbers. Let $n \in \mathbb{N}$. Then the *pidgeon-hole principle* allows us to conclude that there are at least two numbers among our $r_1, \ldots, r_{11}$ which agree in the N^{th} spot. We are now going to prove the statement using the *proof by contradiction* method:

Assume that for any two r_i and r_j of our list above with $i \neq j$, the pair (r_i, r_j) agrees in only finitely many places.

There is only a finite number of ways (55 to be precise) to choose a pair of two different numbers out of the eleven that we are given. Note that if a pair agrees in only finitely many spots, there is a last spot in which it agrees. Hence for each i and j with $i \neq j$ and $1 \leq i \leq 11$ and $1 \leq j \leq 11$, let $N_{i,j}$ be the largest natural number with the property that r_i and r_j agree in the spot $N_{i,j}$. Further, let N_{max} be the maximum of the finite set

$$\{N_{i,j} \mid 1 \leq i, j \leq 11\}.$$

Let M be an arbitrary natural number which is greater than N_{max}, and let i, j be arbitrary different natural numbers between 1 and 11. Then $M > N_{max} \geq N_{i,j}$. Since $N_{i,j}$ is the last spot in which r_i and r_j agree, it follows that they do not agree in the M^{th} spot. Since i and j were picked arbitrarily, we have proved that we may conclude from our assumption that no pair agrees in spot M.

Since M was arbitrarily chosen, we have deduced that for every natural number M larger than N_{max}, there exists no pair which agrees in the M^{th} spot.

In order to lead this last conclusion to a contradiction, we need to find one M which is greater than N_{max} and such that there is a pair that agrees in spot M. Here is such an M:

Set $M = N_{max} + 1$. Since we argued at the very beginning of the proof that for any natural number N there exists a pair which agrees in the N^{th} spot, this is in particular true for $N = M$. Hence the desired contradiction follows and the proof is complete.

Date: October 7, 2006.