SOLUTION FOR PROBLEM SET 4 PROBLEM (4)

For (a), we start with the "$\Rightarrow$" part: Let $f : X \to Y$ be surjective. We need to show that there exists a left inverse $g : Y \to X$ of f. Fix $y \in Y$. Since f is surjective, we know that there exists an $x \in X$ such that $f(x) = y$ (this specific y that we had fixed). Pick such an x, and set

$$g(y) := x.$$

We claim that the map g we just defined is indeed a right inverse to f. To prove this, we need to prove that for any $y \in Y$, the equality $f(g(y)) = y$ holds. Let $y \in Y$ be arbitrary but fixed. Then, by the construction of g above, $g(y)$ is an element of X with the property that $f(g(y)) = y$. This is already what needed to be shown.

Here is the other direction "$\Leftarrow$": Assume that there exists at least one right inverse of f, pick such a right inverse and call it g. We need to prove that f is surjective, i.e., that for any y in Y there exists an $x \in X$ such that $f(x) = y$. Let $y \in Y$ be arbitrary, and set $x = g(y)$. Then x is an element of X, and we have

$$f(x) = f(g(y)) = y,$$

where the first equality follows from the definition of x and the second equality follows because we know that g is a right inverse of f.

QED

For (b), we start with the "$\Rightarrow$" direction: assume that f is bijective, i.e., that it is injective and surjective. In class, we have proved that the injectivity of f implies the existence of a left inverse of f. Pick such a left-inverse and call it g_L. The condition that g_L is a left inverse can be written as

$$g_L \circ f = \mathrm{id}_X,$$

where $\mathrm{id}_X : X \to X$ is the identity map (that means, id_X sends every element x of X to itself). In part (a) we have proved that the surjectivity of f implies the existence of a right inverse of f. Pick such a right inverse and call it g_R. The condition that g_R is a right inverse can be written as

$$f \circ g_R = \mathrm{id}_Y\,.$$

Date: October 8, 2006.

We need to prove that $g_R = g_L$. For this, note that

$$g_L = g_L \circ \mathrm{id}_Y = g_L \circ (f \circ g_R) = (g_L \circ f) \circ g_R = \mathrm{id}_X \circ g_R = g_R.$$

For the "$\Leftarrow$" direction, assume that f possess an inverse g. Then g is at the same time a right and left inverse. From class we know that the existence of a left inverse implies injectivity of f. In (a) we have proved that the existence of a right inverse implies surjectivity of f. Hence f is injective and surjective and thus bijective.

QED