SOLUTION FOR PROBLEM SET 4, PROBLEM (2)

Here is one possibility how to prove it:

Let $\varepsilon > 0$ be an arbitrary but fixed real number. Pick a natural number $N > \frac{1}{\varepsilon}$. Such an N exists by the unboundedness property of the natural numbers. Let n and m be arbitrary natural numbers, both greater or equal to N. Without loss of generality, we assume $m \geq n$. We have

$$\begin{aligned} |a_n - a_m| &= \left|\frac{n+1}{n} - \frac{m+1}{m}\right| \\ &= \left|\frac{m(n+1) - n(m+1)}{nm}\right| \\ &= \left|\frac{m-n}{nm}\right| \\ &= \left|\frac{1}{n} - \frac{1}{m}\right| \\ &= \frac{1}{n} - \frac{1}{m}, \end{aligned}$$

where the last equality follows from $m \geq n$. Further, we have

$$\frac{1}{n} - \frac{1}{m} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon,$$

Where the second inequality follows from $n \geq N$ and the fact that n and N are positive, and the third inequality follows from $N > \frac{1}{\varepsilon}$ and the fact that both sides of this inequality are positive. QED

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