Student name: Daniel Johnston
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QUESTION 1

Tutor's name: Nora Ganter
Tutorial time: 2:15 Tuesday

Algebra assignment 2

1. We will bring M into Smith Normal Form

$$M = \begin{bmatrix} 4 & 2 & 7 \\ 6 & 12 & 0 \\ 2 & 8 & 14 \\ 24 & 21 & 14 \end{bmatrix}$$

$$\xrightarrow{L_0} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 7 \\ 6 & 12 & 0 \\ 2 & 8 & 14 \\ 24 & 21 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 10 & -7 \\ 0 & 18 & -21 \\ 2 & 8 & 14 \\ 24 & 21 & 14 \end{bmatrix} \xrightarrow[R4-12R1]{R3-R1} \begin{bmatrix} 2 & 10 & -7 \\ 0 & 18 & -21 \\ 0 & -2 & 21 \\ 0 & -99 & 98 \end{bmatrix}$$

$$\xrightarrow{C2-5C1} \begin{bmatrix} 2 & 0 & -7 \\ 0 & 18 & -21 \\ 0 & -2 & 21 \\ 0 & -99 & 98 \end{bmatrix}$$

$$\xrightarrow{R_0} \begin{bmatrix} 2 & 0 & -7 \\ 0 & 18 & -21 \\ 0 & -2 & 21 \\ 0 & -99 & 98 \end{bmatrix} \begin{bmatrix} 4 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -21 & 18 & -42 \\ 21 & -2 & 42 \\ 98 & -99 & 196 \end{bmatrix} \xrightarrow[\substack{R3-21R1 \\ R4-98R1}]{R2+21R1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 18 & -42 \\ 0 & -2 & 42 \\ 0 & -99 & 196 \end{bmatrix}$$

$6 = 4 + 2$

$4 = 2 \times 2$

$\therefore \gcd(6,4) = 2$

and $2 = (-1)4 + (1)6$

$\Rightarrow 1 = (-1)(2) + (1)(3)$

Giving us

$$L_0 = \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix}$$

with $\det(L_0) = 1$

$-7 = (-4)2 + 1$

$2 = 2 \times 1$

$\therefore \gcd(-7, 2) = 1$

and $1 = (4)(2) + (1)(-7)$

Giving us

$$R_0 = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$$

with $\det(R_0) = 1$

$18 = (-9)(-2)$

$\therefore \gcd(18, -2) = 2$

and $18 + 8(-2) = 1$

$\Rightarrow (1)(9) + (8)(-1) = 1$

Giving us

$$L_1 = \begin{bmatrix} 1 & 8 \\ 1 & 9 \end{bmatrix}$$

with $\det(L_1) = 1$

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$$\xrightarrow{L_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 18 & -42 \\ 0 & -2 & 42 \\ 0 & -99 & 196 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 294 \\ 0 & 0 & 336 \\ 0 & -99 & 196 \end{bmatrix}$$

$$\xrightarrow{L_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 50 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 99 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 294 \\ 0 & 0 & 336 \\ 0 & -99 & 196 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 14896 \\ 0 & 0 & 336 \\ 0 & 0 & 29498 \end{bmatrix} \xrightarrow{C3 - 14896C1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 336 \\ 0 & 0 & 29498 \end{bmatrix}$$

$$\xrightarrow{L_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 439 & -5 \\ 0 & 0 & -2107 & 24 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 336 \\ 0 & 0 & 29498 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 14 \\ 0 & 0 & 0 \end{bmatrix} = SNF(M) = D$$

$-99 = (-50)2 + 1$

$2 = 2 \times 1$

$\therefore gcd(-99, 2) = 1$

and $1 = (50)2 + (1)(-99)$

giving us

$$L_2 = \begin{bmatrix} 50 & 1 \\ 99 & 2 \end{bmatrix}$$

with $det(L_2) = 1$

$29498 = 87 \times 336 + 266$

$336 = 266 + 70$

$266 = 3 \times 70 + 56$

$70 = 56 + 14$

$56 = 4 \times 14$

$\therefore gcd(336, 29498) = 14$

and,

$14 = 70 - 56$

$= 70 - (266 - 3 \times 70)$

$= -266 + 4 \times 70$

$= -266 + 4 \times (336 - 266)$

$= 4 \times 336 - 5 \times 266$

$= 4 \times 336 - 5 \times (29498 - 87 \times 336)$

$= 439 \times \overset{336}{\cancel{294}} + (-5) \times 29498$

$\Rightarrow (439)(24) + (-5)(2107) = 1$

Giving us

$$L_3 = \begin{bmatrix} 439 & -5 \\ -2107 & 24 \end{bmatrix}$$

with $det(L_3) = 1$

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We now calculate L and R by applying the row and column operations to identity matrices. First we calculate L, so:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{L_0} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R4-12R1]{R3-R1} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 12 & -12 & 0 & 1 \end{bmatrix} \xrightarrow[R3-21R1,\ R4-98R1]{R2+21R1} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -24 & 23 & 0 & 0 \\ 22 & -22 & 1 & 0 \\ 110 & -110 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 & 0 \\ 152 & -153 & 8 & 0 \\ 174 & -175 & 9 & 0 \\ 110 & -110 & 0 & 1 \end{bmatrix} \xrightarrow{L_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 50 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 99 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 152 & -153 & 8 & 0 \\ 174 & -175 & 9 & 0 \\ 110 & -110 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 & 0 \\ 7710 & -7760 & 400 & 1 \\ 174 & -175 & 9 & 0 \\ 15268 & -15367 & 792 & 2 \end{bmatrix} \xrightarrow{L_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 439 & -5 \\ 0 & 0 & -2107 & 24 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 7710 & -7760 & 400 & 1 \\ 174 & -175 & 9 & 0 \\ 15268 & -15367 & 792 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 & 0 \\ 7710 & -7760 & 400 & 1 \\ 46 & 10 & -9 & -10 \\ -186 & -83 & 45 & 48 \end{bmatrix} = L$$

Now, we perform the column operations on a 3x3 identity matrix to find R.

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{C2-5C1} \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_0} \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -5 & 7 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{C3-14896C2} \begin{bmatrix} 4 & -5 & 74487 \\ 0 & 1 & -14896 \\ 1 & 0 & 2 \end{bmatrix} = R$$

Hence, we have algorithmically found invertible matrices L and R and a matrix D in Smith Normal form with $LMR = D$. ✓ 8/8.

END OF QUESTION 1

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QUESTION 2

2.

a)

$$A = \begin{bmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{bmatrix}$$

First, we compute the characteristic polynomial so

$$c(\lambda) = \det(\lambda I - A)$$

$$= \begin{vmatrix} \lambda+2 & -2 & -1 \\ 7 & \lambda-4 & -2 \\ -5 & 0 & \lambda \end{vmatrix}$$

$$= -5 \begin{vmatrix} -2 & -1 \\ \lambda-4 & -2 \end{vmatrix} + \lambda \begin{vmatrix} \lambda+2 & -2 \\ 7 & \lambda-4 \end{vmatrix}$$

$$= -5(4+(\lambda-4)) + \lambda((\lambda+2)(\lambda-4)+14)$$

$$= \lambda(-5+\lambda^2-2\lambda-8+14)$$

$$= \lambda(\lambda^2-2\lambda+1)$$

$$= \lambda(\lambda-1)^2$$

So, the eigenvalues are $\lambda=0$ and $\lambda=1$. We now find the eigenspace for each of these eigenvalues.

$\lambda=0$: $((A-0I)\underset{\sim}{x} = \underset{\sim}{0})$

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ -7 & 4 & 2 & 0 \\ 5 & 0 & 0 & 0 \end{array}\right] \underset{R1\times(-\frac{1}{2})}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -\frac{1}{2} & 0 \\ -7 & 4 & 2 & 0 \\ 5 & 0 & 0 & 0 \end{array}\right] \underset{R3-5R1}{\overset{R2+7R1}{\sim}} \left[\begin{array}{ccc|c} 1 & -1 & -\frac{1}{2} & 0 \\ 0 & -3 & -\frac{3}{2} & 0 \\ 0 & 5 & \frac{5}{2} & 0 \end{array}\right]$$

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$$\begin{matrix} R1\times 2 \\ \sim \\ R2\times(-2) \\ R3\times 2 \end{matrix} \left[\begin{array}{ccc|c} 2 & -2 & -1 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 10 & 5 & 0 \end{array}\right] \underset{R3-\frac{5}{3}R2}{\sim} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \left[\begin{array}{ccc|c} 2 & -2 & -1 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Thus $v_3 = t,\ t \in \mathbb{C}$

$6v_2 = -3v_3$

$\Rightarrow v_2 = -\frac{1}{2}t$

and $v_1 = \dfrac{v_3 + 2v_2}{2}$

$= \dfrac{t - t}{2}$

$= 0$

$\therefore$ The eigenspace for $\lambda = 0$ has dimension 1 and is spanned by the eigenvector $(0, 1, -2)$. $(t = -2)$.

$\underline{\lambda = 1:}$ $(A - I)\underset{\sim}{v} = \underset{\sim}{0}$

$$\left[\begin{array}{ccc|c} -3 & 2 & 1 & 0 \\ -7 & 3 & 2 & 0 \\ 5 & 0 & -1 & 0 \end{array}\right] \underset{R1\times(-\frac{1}{3})}{\sim} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & -\frac{1}{3} & 0 \\ -7 & 3 & 2 & 0 \\ 5 & 0 & -1 & 0 \end{array}\right]$$

$$\begin{matrix} R2+7R1 \\ \sim \\ R3-5R1 \end{matrix} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{10}{3} & \frac{2}{3} & 0 \end{array}\right] \begin{matrix} R1\times 3 \\ R2\times(-3) \\ \sim \\ R3\times 3 \end{matrix} \left[\begin{array}{ccc|c} 3 & -2 & -1 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 10 & 2 & 0 \end{array}\right]$$

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$$\overset{R_3 - 2R_2}{\sim} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \left[\begin{array}{ccc|c} 3 & -2 & -1 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Thus, $v_3 = t, \ t \in \mathbb{C}$

$$v_2 = -\frac{1}{5} v_3 = -\frac{1}{5} t$$

$$v_1 = \frac{2}{3} v_2 + \frac{1}{3} v_3$$

$$= \frac{t}{3} - \frac{2}{15} t$$

$$= \frac{t}{5}$$

$\therefore$ The eigenspace for $\lambda = 1$ has ~~dim~~ dimension and is spanned by the eigenvector $(1, -1, 5)$ $(t = 5)$.

Since the characteristic polynomial has a double root for $\lambda = 1$ and a single root for $\lambda = 0$ there must be two 1's in $JNF(A)$ and one 0. Moreover, since the eigenspace for both eigenvalues has dimension 1, there is only one Jordan block for each eigenvalue. Thus,

$$JNF(A) = J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Since the Jordan block for $\lambda = 1$ has size 2 we will calculate the third vector for the change of basis matrix by computing the generalised eigenvectors satisfying $(A - I)^2 \underset{\sim}{v} = 0$.

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Note that

$$(A-I)^2 = \begin{bmatrix} -3 & 2 & 1 \\ -7 & 3 & 2 \\ 5 & 0 & -1 \end{bmatrix}^2$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 10 & -5 & -3 \\ -20 & 10 & 6 \end{bmatrix}$$

So, $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 10 & -5 & -3 & 0 \\ -20 & 10 & 6 & 0 \end{array}\right] \overset{R3 \leftrightarrow R1}{\sim} \left[\begin{array}{ccc|c} -20 & 10 & 6 & 0 \\ 10 & -5 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\overset{R2+\frac{1}{2}R1}{\sim} \left[\begin{array}{ccc|c} -20 & 10 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R1 \times (-\frac{1}{2})}{\sim} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & \end{array} \\ \left[\begin{array}{ccc|c} 10 & -5 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

Thus, $v_3 = t, \; t \in \mathbb{C}$

$v_2 = s, \; s \in \mathbb{C}$

$$v_1 = \frac{5v_2 + 3v_3}{10}$$

$$= \frac{5s + 3t}{10}$$

how can we guarantee this satisfies $(A-I)v = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$?

Letting $(s,t) = (-1, 5)$ returns the original eigenvector $(1,-1,5)$.
We now use $(s,t) = (2,0)$ to get another linearly independent generalised eigenvector i.e. $(1,2,0)$.
So, all of these eigenvectors can now be used to form the columns of our change of basis B^{-1} matrix, that is:

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$$B^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & -1 \\ -2 & 0 & 5 \end{bmatrix}$$

We now invert B^{-1} to get B. So consider

$$[B^{-1} | I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ -2 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R1 \leftrightarrow R2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ -2 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R3+2R1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & 0 & 2 & 1 \end{array}\right]$$

$$\underset{R3-4R2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -4 & 2 & 1 \end{array}\right]$$

$$\underset{R3\times(-1)}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & -2 & -1 \end{array}\right]$$

$$\underset{R1+R3}{\overset{R2-R3}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -1 & -1 \\ 0 & 1 & 0 & -3 & 2 & 1 \\ 0 & 0 & 1 & 4 & -2 & -1 \end{array}\right]$$

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$$\overset{R1-2R2}{\sim}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -5 & -3 \\ 0 & 1 & 0 & -3 & 2 & 1 \\ 0 & 0 & 1 & 4 & -2 & -1 \end{array}\right]$$

Hence $B = \begin{bmatrix} 10 & -5 & -3 \\ -3 & 2 & 1 \\ 4 & -2 & -1 \end{bmatrix}$

~~then~~ So, we have found matrices such that

$A = B^{-1} J B$ with $J = JNF(A)$.

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b) (i) Over $\mathbb{C}[X]$, where X acts as A we see that

$$X\underset{\sim}{e}_1 = \begin{bmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= -2\underset{\sim}{e}_1 - 7\underset{\sim}{e}_2 + 5\underset{\sim}{e}_3$$

$$\Rightarrow (X+2)\underset{\sim}{e}_1 + 7\underset{\sim}{e}_2 - 5\underset{\sim}{e}_3 = 0 \qquad ①$$

Similarly,

$$X\underset{\sim}{e}_2 = 2\underset{\sim}{e}_1 + 4\underset{\sim}{e}_2$$

$$\Rightarrow -2\underset{\sim}{e}_1 + (X-4)\underset{\sim}{e}_2 = \underset{\sim}{0} \qquad ②$$

i) ???

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and $X\underset{\sim}{e}_3 = \underset{\sim}{e}_1 + 2\underset{\sim}{e}_2$

$$\Rightarrow -\underset{\sim}{e}_1 - 2\underset{\sim}{e}_2 + X\underset{\sim}{e}_3 = \underset{\sim}{0} \qquad ③$$

①, ② and ③ give us relations for the $\underset{\sim}{e}_i$ over $\mathbb{C}[X]$.

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c) Putting these relations into the columns of a matrix gives us T i.e.

$$T = \begin{bmatrix} x+2 & -2 & -1 \\ 7 & x-4 & -2 \\ -5 & 0 & x \end{bmatrix}$$

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d)

$$\begin{bmatrix} x+2 & -2 & -1 \\ 7 & x-4 & -2 \\ -5 & 0 & x \end{bmatrix} \xrightarrow{C1 \leftrightarrow C3} \begin{bmatrix} -1 & -2 & x+2 \\ -2 & x-4 & 7 \\ x & 0 & -5 \end{bmatrix}$$

$$\xrightarrow[R3+xR1]{R2-2R1} \begin{bmatrix} -1 & -2 & x+2 \\ 0 & x & 3-2x \\ 0 & -2x & x^2+2x-5 \end{bmatrix} \xrightarrow[C3+(x+2)C1]{C2-2C1} \begin{bmatrix} -1 & 0 & 0 \\ 0 & x & 3-2x \\ 0 & -2x & x^2+2x-5 \end{bmatrix}$$

$$\xrightarrow{R3+2R2} \begin{bmatrix} -1 & 0 & 0 \\ 0 & x & 3-2x \\ 0 & 0 & x^2-2x+1 \end{bmatrix}$$

$\gcd(x, 3-2x) = 1$ (no comm[on] factors)
and,
$\frac{2}{3}x + \frac{1}{3}(3-2x) = 1$
giving us
$R_0 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ 2x-3 & x \end{bmatrix}^T = \begin{bmatrix} \frac{2}{3} & 2x-[3] \\ \frac{1}{3} & x \end{bmatrix}$
with $\det(R_0) = 1$

$$\xrightarrow{R_0} \begin{bmatrix} -1 & 0 & 0 \\ 0 & x & 3-2x \\ 0 & 0 & (x-1)^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{3} & 2x-3 \\ 0 & \frac{1}{3} & x \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3}(x-1)^2 & x(x-1)^2 \end{bmatrix}$$

$$\xrightarrow{R3-\frac{1}{3}(x-1)^2R2} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x(x-1)^2 \end{bmatrix} = SNF(T) = D$$

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e) Interpreting the matrix D as the sum of submodules gives us

$$V \cong \mathbb{C}[X]/\langle -1\rangle \oplus \mathbb{C}[X]/\langle 1\rangle \oplus \mathbb{C}[X]/\langle x(x-1)^2\rangle$$

$$\cong \mathbb{C}[X]/\langle x(x-1)^2\rangle$$

Since $\mathbb{C}[X]/\langle \pm 1\rangle \cong \mathbb{C}[X]/\mathbb{C}[X]$ which is the trivial group

$$\cong \mathbb{C}[X]/\langle x\rangle \oplus \mathbb{C}[X]/\langle (x-1)^2\rangle$$ (Chinese remainder theorem)

We now treat X as a transformation matrix and show that with respect to some basis, it gives us the Jordan normal form of A.

So, first we restrict our transformation X to the submodule $\mathbb{C}[X]/\langle x\rangle$

Multiplication by X will always return 0 since X satisfies the relation $X=0$ in this submodule. This gives us the Jordan block $[0]$.

We now restrict ourselves to $\mathbb{C}[X]/\langle (x-1)^2\rangle$

We use the basis $\{\bar{1}, \bar{x}-\bar{1}\}$ where $\bar{f} := f + \langle x-1\rangle^2$.

So,

$$X\bar{1} = \bar{x} = \bar{1} + (\bar{x}-\bar{1}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $X(\bar{x}-\bar{1}) = \bar{x}^2 - \bar{x}$

$(\bar{x}-\bar{1})^2 = 0$
$\Rightarrow \bar{x}^2 - 2\bar{x} + \bar{1} = 0$
$\Rightarrow \bar{x}^2 = 2\bar{x} - \bar{1}$

$\vdots$

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$$= (2\bar{X} - \bar{1}) - \bar{X}$$
$$= (\bar{X} - \bar{1})$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus, multiplication by X gives the transformation matrix

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ when we restrict ourselves to $\frac{\mathbb{C}[X]}{\langle (X-1)^2 \rangle}$

Putting these transformation matrices together gives our Jordan Normal Form i.e.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Now, we will discuss the relationship between L and the base change matrix B (or more specifically B^{-1}).

So, consider L

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \frac{1}{3}(2X^2 - X - 10) & -\frac{1}{3}(X^2 - 2X - 5) & 1 \end{bmatrix}$$

In bringing T into Smith Normal Form we made new generators $\underset{\sim}{g_1}, \underset{\sim}{g_2}, \underset{\sim}{g_3}$ that were combinations of the standard basis vectors $\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_3}$. Performing row operations made such changes and the row operations are recorded in L.

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So, reading from the entries of L we have:

$$g_1 = Le_1 = 1e_1 - 2e_2 + \tfrac{1}{3}(2X^2 - X - 10)e_3$$

$$g_2 = Le_2 = 1e_2 - \tfrac{1}{3}(X^2 - 2X - 5)e_3$$

$$g_3 = Le_3 = 1e_3$$

multiply using $X = A$, you should get a B in this way

Hence, we can calculate g_i in terms of the standard basis vectors since X is acting as A.

Now, $-g_1 = 0$

$g_2 = 0$

and $X(X-1)^2 g_3 = 0$ (reading off our matrix D)

Recall that $Xw_1 = 0$, $(X-1)^2 w_2 = 0$ and $(X-1)w_3 = 0$ where w_1, w_2 and w_3 are the eigenvectors (and generalised eigenvector) in the columns of B^{-1}.

If we rewrite our relation for g_3 we obtain:

$$X((X-1)^2 g_3) = 0$$

$$(X-1)(X(X-1)g_3) = 0$$

$$(X-1)^2(Xg_3) = 0$$

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So, $(X-1)^2 \underset{\sim}{g_3} = \underset{\sim}{w_1}$,

$$X \underset{\sim}{g_3} = \underset{\sim}{w_2}$$

and $X(X-1)\underset{\sim}{g_3} = \underset{\sim}{w_3}$

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Thereby providing an explicit relationship between the generators obtained from the ~~rows~~ columns of L and valid eigenvectors that can be used in the columns of the base change matrix B^{-1}.

f)

i) Over the complex numbers, we always require at least three generators (for example, the standard ~~basis~~ basis vectors) to generate $V = \mathbb{C}^3$.

ii) As was shown in part e), we could generate V over $\mathbb{C}[X]$ with the generators $\underset{\sim}{g_1}$, $\underset{\sim}{g_2}$ and $\underset{\sim}{g_3}$ satisfying

$$-\underset{\sim}{g_1} = \underset{\sim}{0}$$

$$\underset{\sim}{g_2} = \underset{\sim}{0}$$

and $X(X-1)^2 \underset{\sim}{g_3} = \underset{\sim}{0}$

However, the first two generators are trivial and not needed, so we can generate V with just one generator over $\mathbb{C}[X]$, namely $\underset{\sim}{g_3}$.

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END OF QUESTION 2

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QUESTION 3

3.

a) $\cos(4\theta) = \cos(2\theta + 2\theta)$

$= \cos^2(2\theta) - \sin^2(2\theta)$ (Addition formula)

$= \cos^2(2\theta) - (1 - \cos^2(2\theta))$ ($\cos^2(\theta) + \sin^2(\theta) = 1$)

$= 2\cos^2(2\theta) - 1$

$= 2\cos^2(\theta + \theta) - 1$

$= 2(\cos^2(\theta) - \sin^2(\theta))^2 - 1$ (Addition formula)

$= 2(\cos^2(\theta) - (1 - \cos^2(\theta)))^2 - 1$ ($\cos^2(\theta) + \sin^2(\theta) = 1$)

$= 2(2\cos^2(\theta) - 1)^2 - 1$

$= 2(4\cos^4(\theta) - 4\cos^2(\theta) + 1) - 1$

$= 8\cos^4(\theta) - 8\cos^2(\theta) + 2 - 1$

$= 8\cos^4(\theta) - 8\cos^2(\theta) + 1$

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Hence,

$\cos(60°) = 8\cos^4(15°) - 8\cos^2(15°) + 1$

$\Rightarrow \frac{1}{2} = 8\cos^4(15°) - 8\cos^2(15°) + 1$ ($\cos(60°) = \frac{1}{2}$)

$\Rightarrow 8\cos^4(15°) - 8\cos^2(15°) + \frac{1}{2} = 0$

$\Rightarrow 8\alpha^4 - 8\alpha^2 + \frac{1}{2} = 0$ ($\alpha = \cos(15°)$)

$\Rightarrow \alpha^4 - \alpha^2 + \frac{1}{16} = 0$

So, let $P(x) = x^4 - x^2 + \frac{1}{16}$

We will first factorise $P(x)$ over $\mathbb{C}$.
So, let $u = x^2$

$\Rightarrow P(u) = u^2 - u + \frac{1}{16}$

$\Rightarrow P(u) = \left(u - \left(\frac{2+\sqrt{3}}{4}\right)\right)\left(u - \left(\frac{2-\sqrt{3}}{4}\right)\right)$

(Quadratic formula, see side working)

$a = 1,\ b = -1,\ c = \frac{1}{16}$

$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{1 \pm \sqrt{1 - \frac{1}{4}}}{2}$

$= \frac{1 \pm \frac{\sqrt{3}}{2}}{2} = \frac{2 \pm \sqrt{3}}{4}$

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$$\Rightarrow p = \left(x - \frac{\sqrt{2+\sqrt{3}}}{2}\right)\left(x + \frac{\sqrt{2+\sqrt{3}}}{2}\right)\left(x - \frac{\sqrt{2-\sqrt{3}}}{2}\right)\left(x + \frac{\sqrt{2-\sqrt{3}}}{2}\right)$$

(Since $u = x^2$)

Since none of these linear factors have rational roots, we cannot reduce $p(x)$ into a cubic factor and linear factor over $\mathbb{Q}$. However, to ensure that $p(x)$ is irreducible over $\mathbb{Q}$ we require that p also can't be split into two quadratic factors (the only other way to reduce a quartic into two factors). So, we will now write down all of the ways to write $p(x)$ as the product of two quadratic polynomials (over $\mathbb{C}$).

So,

$$p(x) = \left(x - \frac{\sqrt{2+\sqrt{3}}}{2}\right)\left(x + \frac{\sqrt{2+\sqrt{3}}}{2}\right)\left(x - \frac{\sqrt{2-\sqrt{3}}}{2}\right)\left(x + \frac{\sqrt{2-\sqrt{3}}}{2}\right)$$

$$= \left(x^2 - \frac{2+\sqrt{3}}{4}\right)\left(x^2 - \frac{2-\sqrt{3}}{4}\right) \quad \text{(First possibility)}$$

$$= \left(x^2 - \left(\frac{\sqrt{2+\sqrt{3}}}{2} + \frac{\sqrt{2-\sqrt{3}}}{2}\right)x + \frac{\sqrt{2+\sqrt{3}}}{2}\frac{\sqrt{2-\sqrt{3}}}{2}\right)\left(x^2 + \left(\frac{\sqrt{2+\sqrt{3}}}{2} + \frac{\sqrt{2-\sqrt{3}}}{2}\right)x + \frac{\sqrt{2+\sqrt{3}}}{2}\frac{\sqrt{2-\sqrt{3}}}{2}\right)$$

(Second possibility)

$$= \left(x^2 + \left(\frac{\sqrt{2-\sqrt{3}}}{2} - \frac{\sqrt{2+\sqrt{3}}}{2}\right)x - \frac{\sqrt{2-\sqrt{3}}}{2}\frac{\sqrt{2+\sqrt{3}}}{2}\right)\left(x^2 + \left(\frac{\sqrt{2+\sqrt{3}}}{2} - \frac{\sqrt{2-\sqrt{3}}}{2}\right)x - \frac{\sqrt{2-\sqrt{3}}}{2}\frac{\sqrt{2+\sqrt{3}}}{2}\right)$$

(Third possibility)

None of these quadratic factors are polynomials over $\mathbb{Q}$. Hence we can conclude that

$$p(\alpha) = \alpha^4 - \alpha^2 + \frac{1}{16}$$

is the irreducible polynomial of α over $\mathbb{Q}$

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b) Letting $u=\alpha^2$ in $p(\alpha)$ gives:

$$p(u) = u^2 - u + \frac{1}{16}$$

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which as shown in part a) has the roots $\frac{2\pm\sqrt{3}}{4}$ over $\mathbb{C}$, neither of which are rational so $p(u)$ must be the irreducible polynomial of u (α^2) over $\mathbb{Q}$ which has degree 2 as required.

c) The roots of $p(\alpha) = \alpha^4 - \alpha^2 + \frac{1}{16}$ were

$$\alpha = \pm\frac{\sqrt{2\pm\sqrt{3}}}{2}$$ as shown in part a).

However, since $\alpha = \cos(15°) > 0$ by definition of $\cos(15°)$ we have that

$$\alpha = \frac{\sqrt{2+\sqrt{3}}}{2}$$ (only real and positive solution)

From this equation we see that α is constructible. This is because the addition and division of constructible numbers is constructible and also the square root of a constructible number is constructible.
More explicitly:

- $2 = 1+1$ is constructible
- $\sqrt{3}$ is constructible
- $2+\sqrt{3}$ is therefore constructible
- $\sqrt{2+\sqrt{3}}$ is therefore constructible
- $\frac{\sqrt{2+\sqrt{3}}}{2}$ is therefore constructible
- α is therefore constructible

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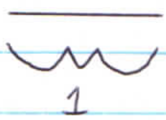
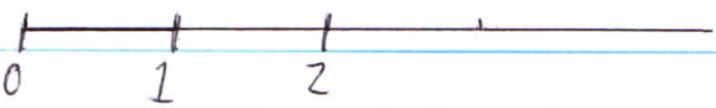
d) $\cos(45°) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

we will show how this is constructible.
So, start with a unit length

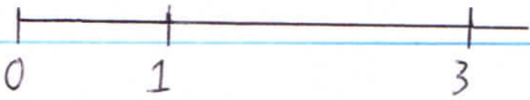


First we measure this length with a compass and then extend the line

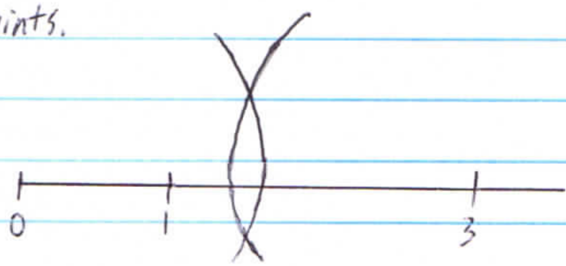
And then, add on another unit as measured by our compass



We have now constructed 2
Next, we repeat this addition procedure to create a length of $1+2=3$.



We now make two arcs with radius $\frac{3}{2} < r < 3$. One of these arcs is centered on 0 whilst the other is centered on 3. We only need to make them large enough so that they intersect at two points.



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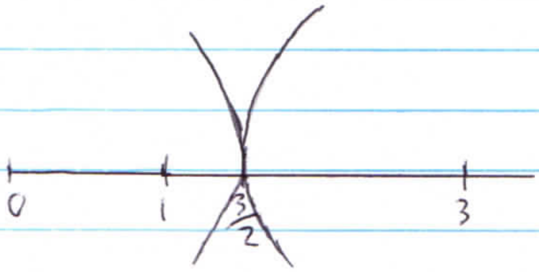
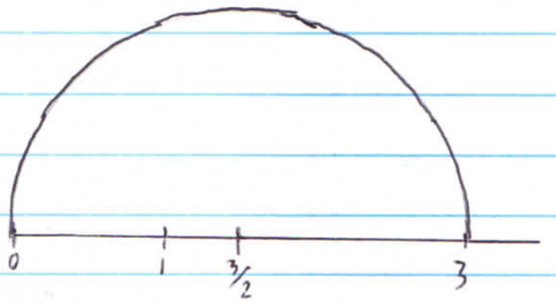
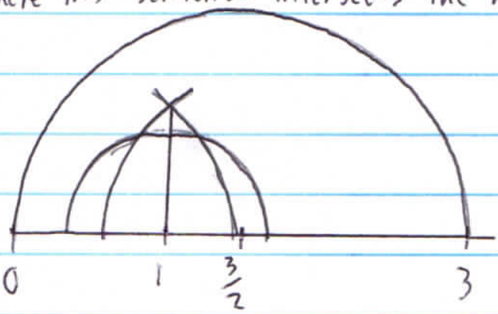
We then join these intersections to the main line to get $\frac{3}{2}$.



Now, we make a semicircle of radius $\frac{3}{2}$ centered on $\frac{3}{2}$.



We now create a semicircle of small radius centred on 1 and then take the points where this semicircle intersects the mainline and create two intersecting arcs as we did when constructing $\frac{3}{2}$.



Joining this intersection to the main line gives us a line perpendicular to the main line at 1. We extend this perpendicular line to our circle of radius $\frac{3}{2}$. We then call the intersection between the perpendicular line and semicircle A and then attach A to 0 and A to 3 creating the diagram on the next page. Note that the points are now assigned letters as shown.

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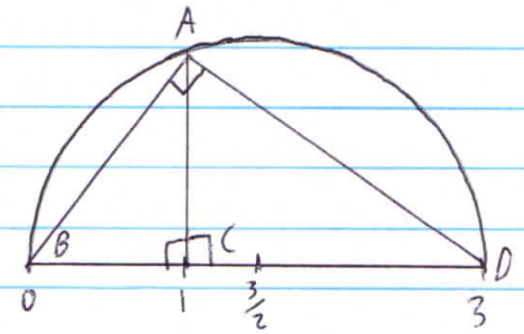

* Note that the angle in a semicircle is $90°$.

Now note that $\Delta ABC \,|||\, \Delta ABD$ since

$$\angle BCA = \angle BAD = 90°$$

and $\angle ABC$ is shared (two pairs of identical angles entail similarity)

Moreover $\Delta ACD \,|||\, \Delta ABD$ since

$$\angle ACD = \angle BAD = 90°$$

and $\angle ADB$ is shared (two pairs of identical angles entail similarity)

Hence $\Delta ABC \,|||\, \Delta ABD \,|||\, \Delta ACD$

$$\Rightarrow \Delta ABC \,|||\, \Delta ACD$$

So, we have:

$$\frac{BC}{AC} = \frac{AC}{CD}$$ (ratios of sides are preserved in similar triangles)

$$\Rightarrow (AC)^2 = BC \times CD$$
$$\Rightarrow (AC)^2 = 1 \times 2$$

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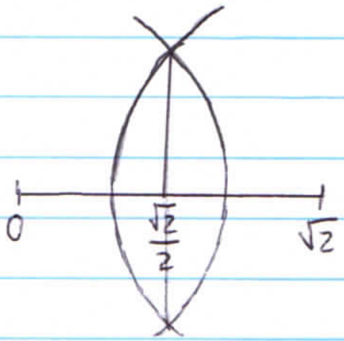
$\Rightarrow AC = \sqrt{2}$

Thus, we have constructed $\sqrt{2}$.

Now, we use the same intersecting arc method to halve $\sqrt{2}$ as we did with $\frac{3}{2}$.



Thus we have constructed $\frac{\sqrt{2}}{2} = \cos(45^\circ)$ as required.

Now, $\cos(3\theta)$

$= Re(e^{i3\theta})$

$= Re((\cos(\theta) + i\sin(\theta))^3)$

$= Re(\cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta))$

$= \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$

$= \cos^3(\theta) - 3\cos(\theta)(1 - \cos^2(\theta))$ $\quad (\cos^2(\theta) + \sin^2(\theta) = 1)$

$= 4\cos^3(\theta) - 3\cos(\theta)$

Thus, $\cos(45^\circ) = 4\cos^3(15^\circ) - 3\cos(15^\circ)$

$\Rightarrow \cos(45^\circ) = 4\alpha^3 - 3\alpha$

$\Rightarrow 4\alpha^3 - 3\alpha - \cos(45^\circ) = 0$

Let $f(x) = 4x^3 - 3x - \cos(45^\circ)$

we see that

$f(-\cos(45^\circ)) = 4(-\cos^3(45^\circ)) + 3\cos(45^\circ) - \cos(45^\circ)$

$= 2\cos(45^\circ) - 4\cos^3(45^\circ)$

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$$= \frac{2}{\sqrt{2}} - 4\left(\frac{1}{\sqrt{2}}\right)^3$$

$$= \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}}$$

$$= 0$$

$\therefore -\cos(45°) \in \mathbb{Q}[\cos(45°)]$ is a root of $f(x)$ so we will divide f by $x + \cos(45°)$ to see if we can produce the irreducible polynomial of $\cos(15°)$ over $\cos(45°)$.

So,

$$\begin{array}{r|l} & 4x^2 - 4x\cos(45°) - 1 \\ \hline x + \cos(45°) & 4x^3 + 0x^2 - 3x - \cos(45°) \\ & \underline{4x^3 + 4x^2\cos(45°)} \\ & -4x^2\cos(45°) - 3x \\ & \underline{-4x^2\cos(45°) - 2x} \\ & -x - \cos(45°) \\ & \underline{-x - \cos(45°)} \\ & 0 \end{array}$$

$\cos^2(45°) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

$\therefore f(x) = (x + \cos(45°))(4x^2 - 4\cos(45°)x - 1)$

So, we now consider $g(x) = 4x^2 - 4\cos(45°)x - 1$

Note that $\mathbb{Q}[\cos(45°)] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$

Since $\cos(45°) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and is a root of the quadratic polynomial $q(x) = x^2 - \frac{1}{2}$ which is irreducible over $\mathbb{Q}$ (its only roots over $\mathbb{C}$ are $\pm\frac{1}{\sqrt{2}}$ which are both irrational).

We first solve $g(x) = 0$ over $\mathbb{C}$ to give

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(Quadratic formula)}$$

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$$= \frac{4\cos(45^\circ) \pm \sqrt{16\cos^2(45^\circ) - 4\times4\times(-1)}}{2\times4}$$

$$= \frac{4\cos(45^\circ) \pm \sqrt{8+16}}{8} \qquad \left(\cos^2(45^\circ) = \left(\tfrac{1}{\sqrt{2}}\right)^2 = \tfrac{1}{2}\right)$$

$$= \frac{2\cos(45^\circ) \pm \sqrt{6}}{4}$$

$$= \frac{\cancel{2}\cos(45^\circ) \pm 2\sqrt{3}\times\frac{\sqrt{2}}{2}}{\cancel{4}\,2}$$

$$= \frac{\cos(45^\circ) \pm \sqrt{3}\cos(45^\circ)}{2}$$

$\notin \mathbb{Q}[\cos(45^\circ)]$

since $\sqrt{3} \notin \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$

Thus $g(\alpha) = 4\alpha^2 - 4\cos(45^\circ)\alpha - 1$ is irreducible over $\mathbb{Q}[\cos(45^\circ)]$.

i.e. $h(\alpha) = \alpha^2 - \cos(45^\circ)\alpha - \frac{1}{4}$ is the irreducible polynomial of $\alpha = \cos(15^\circ)$ over $\mathbb{Q}[\cos(45^\circ)]$ ($h(\alpha) = \frac{g(\alpha)}{4}$ to obtain a monic polynomial with the same roots)

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e) Our irreducible polynomial for $\cos(15°)$ over $\mathbb{Q}[\cos(45°)]$ is:

$$h(\alpha) = \alpha^2 - \cos(45°) - \frac{1}{4}$$

In part d), we showed that the roots of this equation were

$$\alpha = \frac{\cos(45°) \pm \sqrt{3}\cos(45°)}{2}$$

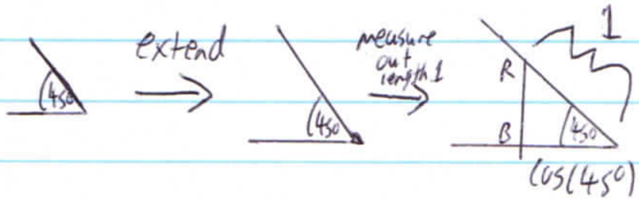
However, since $\alpha = \cos(15°) > 0$ we take the positive solution to get

$$\alpha = \frac{\cos(45°) + \sqrt{3}\cos(45°)}{2}$$

$$= \frac{(1+\sqrt{3})\cos(45°)}{2}$$

We will construct this number step by step. However, we start with an already constructed 45° angle.

This gives us $\cos(45°)$ since if we extend the lines that define the angle and measure out a distance of 1 for the hypotenense, then its projection onto the other line gives us $\cos(45°)$ as shown below.



Note that to project the point labelled R onto the point labelled B we first draw arcs of the same radius (or just a semicircle) ~~circle~~ centred on R intersecting the horizontal line at two points.

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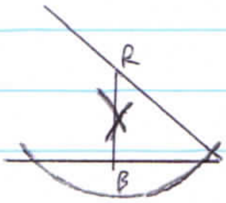
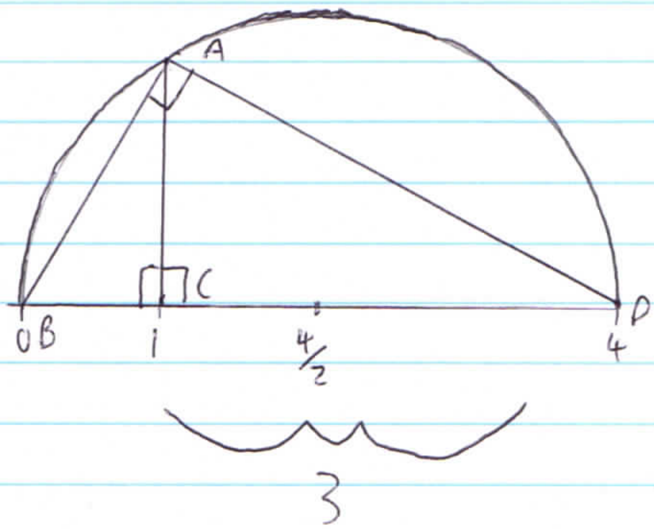
Afterwards, we then construct two arcs centred on these intersection points and their intersection is then collinear with R and B allowing us to construct the projection RB.

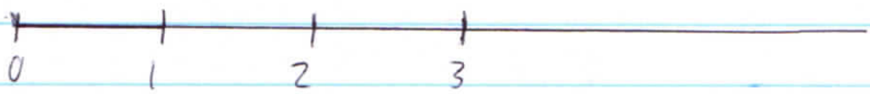


Now that we have $\cos(45°)$ we now construct $\sqrt{3}$. To do so, we create an identical diagram to the one used to construct $\sqrt{2}$ in part d). The only difference is that we have a base partitioned into 1 and 3 but the algorithm is exactly the same, giving us $\sqrt{3}$.



$AC = \sqrt{3}$

Note that the construction of 3 used in this diagram is just an addition of 3 unit lengths that we can construct on a line of arbitrary length.



Now that we've constructed $\sqrt{3}$, we can easily construct $\sqrt{3}+1$ by adding a unit length to $\sqrt{3}$ on a line.

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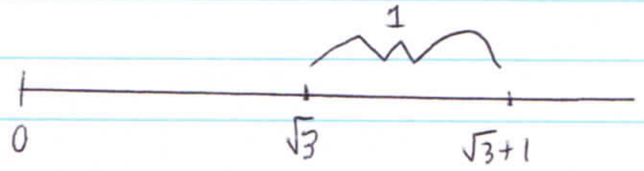
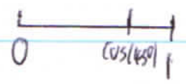
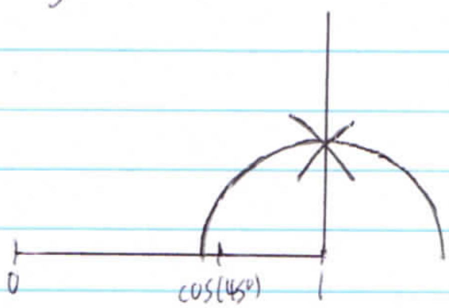


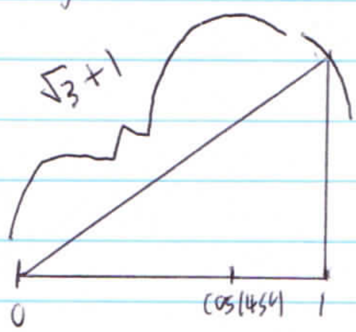
We will now attempt to multiply $1+\sqrt{3}$ and $\cos(45°)$. So, first we construct a line with $\cos(45°)$ measured out.



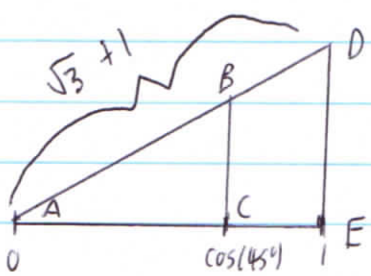
We now construct a line perpendicular to 1 using intersecting arcs as we did during our construction of $\sqrt{2}$ in part d).



We now draw an arc centred on 0 that intersects this perpendicular line and join the intersection point to 0.



Finally we draw a perpendicular line at $\cos(45°)$ (as we did at 1) and label the points as shown



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Note that $\Delta ABC \;|||\; \Delta ADE$ since

$$\angle ACB = \angle AED = 90^\circ$$

and $\angle DAE$ is shared (two pairs of identical angles entail similarity)

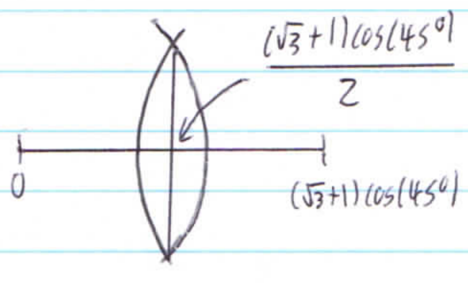
Hence $\frac{AC}{AE} = \frac{AB}{AD}$ (ratios are preserved in similar triangles)

$$\Rightarrow \frac{\cos(45^\circ)}{1} = \frac{AB}{(\sqrt{3}+1)}$$

$$\Rightarrow AB = (\sqrt{3}+1)\cos(45^\circ)$$

So, we have now constructed $(\sqrt{3}+1)\cos(45^\circ)$. Finally, we halve this length to get $\cos(15^\circ) = \frac{(\sqrt{3}+1)\cos(45^\circ)}{2}$.

We do this using our typical halving method by measuring out $(\sqrt{3}+1)\cos$ and drawing intersecting arcs of equal radius at 0 and $(\sqrt{3}+1)\cos(45^\circ)$ and connecting the two intersections.



Now that we have constructed $\cos(15^\circ)$ we can produce a triangle with angle 15°. We do this by starting with $\cos(15^\circ)$ and constructing a perpendicular line (as we did before). Then we connect this perpendicular

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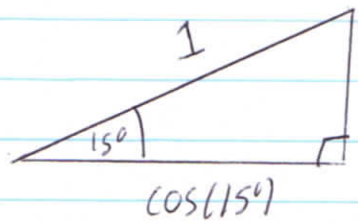
line to the origin with a length of 1 (by finding the intersection between a circle of radius 1 centred on the origin and the perpendicular line and joining the intersection to 01.



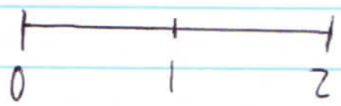
$\cos(\theta) = \frac{a}{h}$

So, we have successfully constructed 15° from a 45° angle.

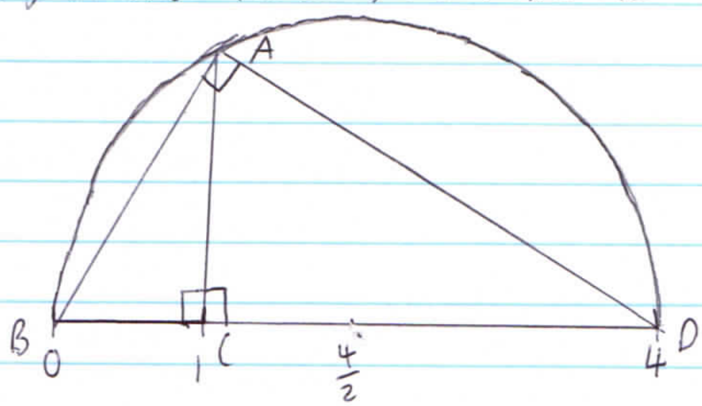
f) Our expression for $\cos(15°)$ obtained in parts (a)-(c) was

$$\cos(15°) = \alpha = \frac{\sqrt{2+\sqrt{3}}}{2}$$

We will construct this from scratch.
So, first construct 2 by adding together two unit lengths on a line.



Then we construct $\sqrt{3}$ using the same method that we used in part e.
The diagram below reiterates the method used



$AC = \sqrt{3}$

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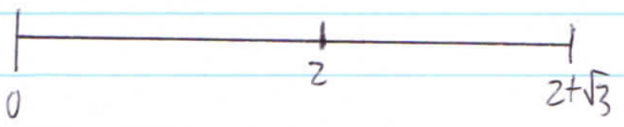
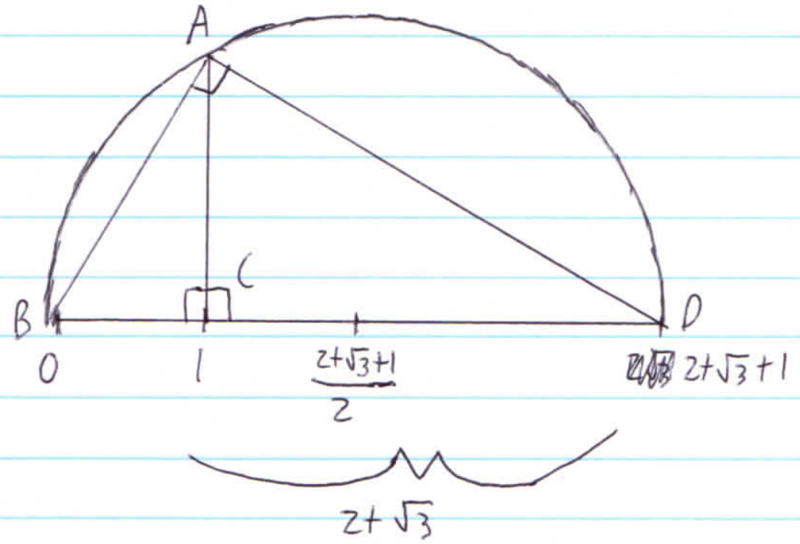
Now, we construct $2+\sqrt{3}$ by adding together our constructed 2 and $\sqrt{3}$ on a line

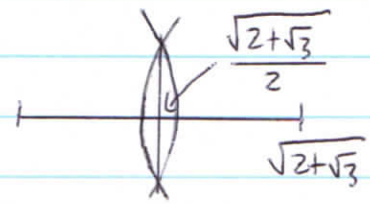


We can then construct $\sqrt{2+\sqrt{3}}$ using the same square root algorithm used for $\sqrt{3}$ but with $CD = 2+\sqrt{3}$ instead of 3.



In this case $AC = \sqrt{2+\sqrt{3}}$

Finally we halve this number to get $\cos(15^\circ) = \frac{\sqrt{2+\sqrt{3}}}{2}$. We do this using the method of intersecting arcs that we have used multiple times already. This is illustrated below.



Now that we have $\cos(15^\circ)$ we can construct 15° using the same triangle from part d) i.e.

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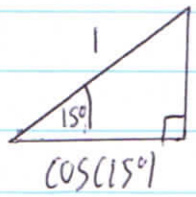

Thus, we have constructed a 15° angle from scratch.

g)

(1) $\mathbb{Q}[\cos(45°)] \cong \mathbb{Q}[X] / \langle X^2 - \frac{1}{2} \rangle$

Since $\cos(45°) = \frac{1}{\sqrt{2}}$ and thus satisfies $X^2 - \frac{1}{2} = 0$ and $X^2 - \frac{1}{2}$ is irreducible over $\mathbb{Q}$ since its only roots over $\mathbb{C}$ are irrational $(\pm \frac{1}{\sqrt{2}})$.

(2) $\mathbb{Q}[\cos(15°)] \cong \mathbb{Q}[X] / \langle X^4 - X^2 + \frac{1}{16} \rangle$

Using our result from part a)

(3) $\mathbb{Q}[\cos^2(15°)] \cong \mathbb{Q}[X] / \langle X^2 - X + \frac{1}{16} \rangle$

Using our result from part b)

(4) $\mathbb{Q}[\cos(15°)] \cong \mathbb{Q}[\cos(45°)][X] / \langle X^2 - \cos(45°) - \frac{1}{4} \rangle$

Using our result from part d)

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(5)

$$\cos(15^\circ) = \frac{(1+\sqrt{3})\cos(45^\circ)}{2} = \frac{\sqrt{6}+\sqrt{2}}{4} \qquad \text{(from part d))}$$

$$\Rightarrow \cos^2(15^\circ) = \frac{6+2\sqrt{3}+2}{16}$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{8}$$

So, $\mathbb{Q}[\cos^2(15^\circ)] = \{a + b\sqrt{3} : a, b \in \mathbb{Q}\}$

Since $\sqrt{3}$ is the only irrational component of $\cos^2(15^\circ)$.

Consider the polynomial $p(X) = X^2 - \cos^2(15^\circ)$ over $\mathbb{Q}[\cos^2(15^\circ)]$. This is irreducible since its two roots (over $\mathbb{C}$) are $\pm\cos(15^\circ)$ neither of which are elements of $\mathbb{Q}[\cos^2(15^\circ)]$ since $\cos(15^\circ)$ contains a $\sqrt{2}$ term and $\sqrt{2} \notin \mathbb{Q}[\cos^2(15^\circ)]$.

Hence,

$$\mathbb{Q}[\cos(15^\circ)] = \mathbb{Q}[\cos^2(15^\circ)][X] \big/ \langle X^2 - \cos^2(15^\circ)\rangle$$

END OF QUESTION 3

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QUESTION 4

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4.

a) $\mathbb{F}_9$ is the unique splitting field of the polynomial X^9-X over $\mathbb{F}_3$. We will first factorise this polynomial over $\mathbb{F}_3$ to be able to determine $\mathbb{F}_9$.

So,

$$X^9-X = X(X^8-1)$$
$$= X(X-1)(X^7+X^6+X^5+X^4+X^3+X^2+X+1) \quad \text{(Geometric series formula)}$$

Since $\mathbb{F}_9$ is an algebraic extension of $\mathbb{F}_3$ we also expect $(X-2)$ to be a factor.

Now,

$$\begin{array}{r|l}
 & X^6+0X^5+X^4+0X^3+X^2+0X+1 \\
\hline
X-2 & X^7+X^6+X^5+X^4+X^3+X^2+X+1 \\
 & \underline{X^7-2X^6} \\
 & 0X^6+X^5 \\
 & \underline{0X^6+0X^5} \\
 & X^5+X^4 \\
 & \underline{X^5-2X^4} \\
 & 0X^4+X^3 \\
 & \underline{0X^4+0X^3} \\
 & X^3+X^2 \\
 & \underline{X^3-2X^2} \\
 & 0X^2+X \\
 & \underline{0X^2+0X} \\
 & X+1 \\
 & \underline{X-2} \\
 & 0
\end{array}$$

Hence, $X^9-X = X(X-1)(X-2)(X^6+X^4+X^2+1)$

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We expect the remaining irreducible factors to be quadratic since they will give us the other elements of $\mathbb{F}_9$ and $[\mathbb{F}_9 : \mathbb{F}_3] = 2$.

We first consider the reducible quadratic polynomials over $\mathbb{F}_3$ i.e.

- X^2
- $(X-1)^2 = X^2 - 2X + 1 = X^2 + X + 1$
- $(X-2)^2 = X^2 + 2X + 1$
- $X(X-1) = X^2 - X = X^2 + 2X$
- $X(X-2) = X^2 - 2X = X^2 + X$
- $(X-1)(X-2) = X^2 + 2$

So, the remaining irreducible polynomials are
X^2+1, X^2+X+2 and X^2+2X+2
and we see that

$$
\begin{aligned}
&(X^2+1)(X^2+X+2)(X^2+2X+2) \\
&= (X^4+X^3+2X^2+X^2+X+2)(X^2+2X+2) \\
&= (X^4+X^3+X+2)(X^2+2X+2) \\
&= (X^6+2X^5+2X^4+X^5+2X^4+2X^3+X^3+2X^2+2X+2X^2+X+1) \\
&= X^6+X^4+X^2+1
\end{aligned}
$$

Therefore,

$$X^9 - X = X(X-1)(X-2)(X^2+1)(X^2+X+2)(X^2+2X+2)$$

Out of the irreducible quadratic factors X^2+1 is "easier" to work with so we will use it to find $\mathbb{F}_9$. Since X^2+1 is irreducible and thus maximal we have that

$$\mathbb{F}_9 \cong \mathbb{F}_3[X] / \langle X^2+1 \rangle$$

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Using this isomorphism we can construct the following addition and multiplication tables for $\mathbb{F}_9$. Note that $\alpha := \bar{x} := x + \langle x^2+1 \rangle$ is a root of the polynomial x^2+1 over $\mathbb{F}_9$.

$+$	0	1	2	α	2α	$\alpha+1$	$2\alpha+1$	$\alpha+2$	$2\alpha+2$
0	0	1	2	α	2α	$\alpha+1$	$2\alpha+1$	$\alpha+2$	$2\alpha+2$
1	1	2	0	$\alpha+1$	$2\alpha+1$	$\alpha+2$	$2\alpha+2$	α	2α
2	2	0	1	$\alpha+2$	$2\alpha+2$	α	2α	$\alpha+1$	$2\alpha+1$
α	α	$\alpha+1$	$\alpha+2$	2α	0	$2\alpha+1$	1	$2\alpha+2$	2
2α	2α	$2\alpha+1$	$2\alpha+2$	0	α	1	$\alpha+1$	2	$\alpha+2$
$\alpha+1$	$\alpha+1$	$\alpha+2$	α	$2\alpha+1$	1	$2\alpha+2$	2	2α	0
$2\alpha+1$	$2\alpha+1$	$2\alpha+2$	2α	1	$\alpha+1$	2	$\alpha+2$	0	α
$\alpha+2$	$\alpha+2$	α	$\alpha+1$	$2\alpha+2$	2	2α	0	$2\alpha+1$	1
$2\alpha+2$	$2\alpha+2$	2α	$2\alpha+1$	2	$\alpha+2$	0	α	1	$\alpha+1$

$\cdot$	0	1	2	α	2α	$\alpha+1$	$2\alpha+1$	$\alpha+2$	$2\alpha+2$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	α	2α	$\alpha+1$	$2\alpha+1$	$\alpha+2$	$2\alpha+2$
2	0	2	1	2α	α	$2\alpha+2$	$\alpha+2$	$2\alpha+1$	$\alpha+1$
α	0	α	2α	2	1	$\alpha+2$	$\alpha+1$	$2\alpha+2$	$2\alpha+1$
2α	0	2α	α	1	2	$2\alpha+1$	$2\alpha+2$	$\alpha+1$	$\alpha+2$
$\alpha+1$	0	$\alpha+1$	$2\alpha+2$	$\alpha+2$	$2\alpha+1$	2α	2	1	α
$2\alpha+1$	0	$2\alpha+1$	$\alpha+2$	$\alpha+1$	$2\alpha+2$	2	α	2α	1
$\alpha+2$	0	$\alpha+2$	$2\alpha+1$	$2\alpha+2$	$\alpha+1$	1	2α	α	2
$2\alpha+2$	0	$2\alpha+2$	$\alpha+1$	$2\alpha+1$	$\alpha+2$	α	1	2	2α

b) We will construct $GL_2(\mathbb{F}_3)$ row by row to determine its size.

In row 1 we can have any element of $\mathbb{F}_3^2$ except for $(0,0)$ since this always causes the matrix to be singular. So we have a total of $3^2-1=8$ possibilities.

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In row 2 we can have any element of $\mathbb{F}_3^2$ except $(0,0)$ or a multiple of the first row as this would also cause the matrix to be singular. There are 2 non-zero multiples of any element of $\mathbb{F}_3$ (i.e. multiplication by 1 or 2) giving us a total of $(3^2-1-2)=6$ possibilities for row 2.

Multiplying these permutations for row 1 and row 2 gives

$$\# GL_2(\mathbb{F}_3) = 8 \times 6 = 48$$

c) We are treating $\{1, \alpha\}$ as a basis for $\mathbb{F}_9$. So, in considering the field automorphisms of $\mathbb{F}_9$ we need to only consider where to map 1 and α whilst preserving their relations. Let $\varphi: \mathbb{F}_9 \to \mathbb{F}_9$ be a field automorphism of $\mathbb{F}_9$. Firstly $\varphi(1)=1$ by the definition of a field homomorphism so we always map the basis element 1 to itself. To consider where to map α, we find the other root of the quadratic polynomial x^2+1 which would have the same relations as α. By testing our other elements, it is clear that 2α is the other root since

$$(2\alpha)^2+1 = 2+1 = 0$$

Nice

(Using our multiplication and addition tables for $\mathbb{F}_9$)

So, α can either be mapped to α or 2α. The first of these automorphisms has the matrix representation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$ the identity automorphism

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Whilst the other automorphism has the representation

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

i.e. $1 \to 1$
$\alpha \to 2\alpha$

Putting these matrices together gives us the isomorphism

$$Aut(\mathbb{F}_9) \cong \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right\rangle \subseteq GL_2(\mathbb{F}_3)$$

equipped with the operation of matrix multiplication. Note that this is a subgroup of order 2 as we expect since there are only two automorphisms.

END OF QUESTION 4