Assignment 1

Due Monday, 3 April 2017, Noon

1. Let $N \trianglelefteq G$ be a normal subgroup, and assume that there exists a group homomorphism $p : G \to N$ that is left inverse to the inclusion of N in G.

   (a) Show that there exists a group homomorphism

   $$f : G \longrightarrow N \times (G/N)$$

   satisfying

   $$(\forall n \in N) \qquad f(n) = (1, n)$$

   and

   $$(\forall g \in G) \qquad (pr_2 \circ f)(g) = gN.$$

   Here pr_2 denotes the projection to the second factor.

   (b) Prove that the homomorphism f you constructed in part (a) is in fact an isomorphism.

   (c) Reformulate the questions and answers above in terms of short exact sequences.

2. Let $\mathbb{Z}[i]$ be the ring of Gaussian integers, and $\mathbb{Z}[i, j, k]$ the ring of integer quaternions.

   (a) Find all ring homomorphisms

   i. from $\mathbb{Z}[i]$ to $\mathbb{Z}[i]$,

   ii. from $\mathbb{Z}[i]$ to $\mathbb{Z}[i, j, k]$,

   iii. from $\mathbb{Z}[i, j, k]$ to $\mathbb{Z}[i]$, and

   iv. from $\mathbb{Z}[i, j, k]$ to $\mathbb{Z}[i, j, k]$.

   (b) For each of your answers in (a) indicate which are endomorphisms, which are automorphisms, what are the respective kernels and images.

3. Let A be the abelian group

   $$A = (\mathbb{Z}/4\mathbb{Z})^3.$$

(a) Identify the endomorphism ring of A.

(b) Identify all possible ways to equip A with the structure of a module over the group algebra $\mathbb{Z}[S_3]$. Justify your answer.

4. In this question $\mathbb{Z}$ refers to the abelian group $(\mathbb{Z}, +)$.

(a) Identify the group of (group) automorphisms of $\mathbb{Z}$.

(b) Find all possible actions of $\mathbb{Z}$ on itself.

(c) For each action ϱ you found in (b), form the semi-direct product $\mathbb{Z} \rtimes_\varrho \mathbb{Z}$ and describe it in terms of generators and relations.

(d) For each of these actions write $\mathbb{Z} \rtimes_\varrho \mathbb{Z}$ as a quotient of a free group.

Justify your answers.