Assignment 2

Due Friday, 26 May 2017, Noon

1. Let M be the matrix with integer coefficients

$$M = \begin{bmatrix} 4 & 2 & 7 \\ 6 & 12 & 0 \\ 2 & 8 & 14 \\ 24 & 21 & 14 \end{bmatrix}.$$

Use the algorith from class (Wikipedia) to bring M into Smith Normal Form, i.e., find invertible matrices L and R and a matrix D in Smith Normal Form with $LMR = D$. Document your steps.

2. Let A be the matrix

$$A = \begin{bmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{bmatrix}$$

acting on the complex vector space $V = \mathbb{C}^3$.

(a) Use the method you learned in *Group Theory and Linear Algebra* to write A in the form

$$A = B^{-1}JB$$

with J in Jordan normal form. Show your work. Along the way, you find the characteristic polynomial of A. Record this, too.

(b) Consider now the standard basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ of $\mathbb{C}^3$. As in the proof of the Jorndan Normal Form Theorem in class, view $\mathbb{C}^3$ as a module over $\mathbb{C}[X]$, where X acts as A. What relations do the $\vec{e}_i$ satisfy

i. over $\mathbb{C}$

ii. over $\mathbb{C}[X]$?

(c) Write down the matrix T with entries from $\mathbb{C}[X]$ that encodes these generators and relations, as we did in the proof of the normal form.

(d) Now bring T into Smith Normal Form, documenting each step and keeping track also of the matrices L and R, where

$$LTR = D.$$

(e) Work out the details of how the equation $LTR = D$ encodes the Jordan Normal Form for A. In particular, what was the meaning of R, and how does it relate to the base change matrix B of part (a)?

(f) How many generators (at least) to you need to generate V as a module over

i. the complex numbers?

ii. the polynomial ring $\mathbb{C}[X]$?

3. Let $\alpha = \cos(15°)$.

(a) Using the addition formulas for sine and cosine, find the irreducible polynomial of α over $\mathbb{Q}$.

(b) Show that the irreducible polynomial for α^2 over $\mathbb{Q}$ has degree two.

(c) From here, argue that $\cos(15°)$ is constructible.

(d) Assume now that $\cos(45°)$ has already been constructed. (How?) Find the irreducible polynomial of $\cos(15°)$ over $\mathbb{Q}[\cos(45°)]$.

(e) Translate (d) into a step by step construction, using straight-edge and compass, of a $15°$ angle from an already constructed $45°$ angle.

(f) Translate (a) – (c) into a construction of a $15°$ angle from scratch (starting with two arbitrary distinct points in the plane).

(g) Write each of the above field extensions

$$\begin{aligned}
\mathbb{Q} &\subset \mathbb{Q}[cos(45°)] && (1)\\
\mathbb{Q} &\subset \mathbb{Q}[cos(15°)] && (2)\\
\mathbb{Q} &\subset \mathbb{Q}[cos^2(15°)] && (3)\\
\mathbb{Q}[cos(45°)] &\subset \mathbb{Q}[cos(15°)] && (4)\\
\mathbb{Q}[cos^2(15°)] &\subset \mathbb{Q}[cos(15°)] && (5)
\end{aligned}$$

as quotient of the polynomial ring over the smaller field.

4. (a) Working directly from the definition as splitting field over $\mathbb{F}_3$, work out the addition and multiplication tables of $\mathbb{F}_9$.

   (b) Let $GL_2(\mathbb{F}_3)$ be the group of invertible 2×2 matrices with entries from $\mathbb{F}_3$. How many elements does $GL_2(\mathbb{F}_3)$ have?

   (c) Identify the subgroup of $GL_2(\mathbb{F}_3)$ consisting of the field automorphisms of $\mathbb{F}_9$.