Solutions for Tutorial 1

Topic 1: Semi-direct products

1. Show that the tetrahedral group is isomorphic to the semi-direct product of the Klein four group and a cyclic group of order three:
$$T \cong K_4 \rtimes (\mathbb{Z}/3\mathbb{Z}).$$

2. Show further that the tetrahedral group is isomorphic to the alternating group on four elements:
$$T \cong A_4.$$

Proof of 1 and 2. Let V be the set of vertices of the regular tetrahedron. Labeling the vertices, we obtain a bijection
$$V \cong \{1, 2, 3, 4\}.$$
The action of the tetrahedral group on V is encoded in an injective map
$$T \hookrightarrow Bij(V, V) \cong S_4.$$
Indeed, any symmetry of the tetrahedron is uniquely determined by its effect on V, hence injectivity. To determine the number of elements of T, we apply the orbit stabilizer theorem to the first vertex. Its orbit is all of V, while its stabilizer consists of three rotations. Hence the order of T equals $3 \cdot 4 = 12$. The elements of order two of T are exactly the rotations through the midpoints of opposite edges. There are three of them, corresponding to the permutations (12)(34), (13)(24) and (14)(23). Together with the identity, these form a subgroup N of A_4 that is isomorphic to the Klein four group. All the other elements of T are rotations of order 3 around the axis fixing some vertex of T. They are organized into three cosets of N as follows:
$$N \qquad \textit{and} \qquad \{(123), (134), (243), (142)\} \qquad \textit{and} \qquad \{(132), (143), (234), (124)\}.$$
Together, these form the alternating group A_4. N is normal, since the cycle decomposition is invariant under conjugation of permutations. An example of a system of representatives for N is
$$\mathcal{R} = \{e, (123), (132)\}.$$
This choice of $\mathcal{R}$ is in fact a subgroup of A_4, which is cyclic of order 3. This proves that A_4 is isomorphic to a semi-direct product as claimed in 1. For the sake of completeness, we will also identify the action of $\mathbb{Z}/3\mathbb{Z} \cong \langle (123) \rangle$ on $K_4 \cong \{e, (12)(34), (13)(24), (14)(23)\}$. It is enough to identify the action of the generator. This needs to fix the identity element e, and it permutes the other three elements as follows:
$$(12)(34) \longmapsto (14)(23) \longmapsto (13)(24) \longmapsto (12)(34).$$

□

3. Show that the octahedral group is isomorphic to the semi-direct product of the Klein four group and the symmetric group on three elements:

$$O \;\cong\; K_4 \rtimes S_3.$$

4. Show further that the octahedral group is isomorphic to the symmetric group on four elements:

$$O \;\cong\; S_4.$$

Proof of 3. and 4. The mid-points of the faces of the regular octahedron form the vertices of a cube and vice versa, so that O is isomorphic to the symmetry group of the cube. Let us assume that we have labeled the faces of the cube as is customary for dice, i.e., ⚀ ⚁ ⚂ ⚃ ⚄ ⚅ with opposite faces adding up to seven. The orbit of the face ⚀ has six elements, its stabilizer consists of four rotations, so the order of O equals 24. This time the map

$$O \longrightarrow S_4$$

is given by the action of O on the four long diagonals of the cube. Lets number them as follows: diagonal 1 connects the vertex (⚀⚁⚂) with (⚅⚄⚃), diagonal 2 connects (⚀⚂⚄) and (⚅⚃⚁), diagonal 3 connects (⚀⚄⚃) and (⚅⚁⚂) and diagonal 4 connects (⚀⚃⚁) and (⚅⚂⚄). Then the rotations fixing the faces ⚀ and ⚅ form the subgroup $\langle(1234)\rangle$, the $180°$ rotation through the midpoints of the edges (⚁⚂) and (⚄⚃) is the transposition (13), the rotations around the vertex (⚀⚁⚂) form the subgroup $\langle(234)\rangle$, and so on. Inside S_4, the subgroup N is still normal (for the same reason). There are now three more cosets:

$$\{(12),(34),(1324),(1423)\}$$

and

$$\{(13),(1234),(24),(1432)\}$$

and

$$\{(23),(1342),(1243),(14)\}.$$

A system of representatives is now given by

$$\mathcal{R} = \{e,(123),(132),(12),(13),(23)\},$$

which is isomorphic to S_3, acting on the three non-trivial elements of the four group N by permutations, e.g., (12) swaps (13)(24)) with (14)(23), etc. □

Topic 2: Central extensions

Assume we are given a short exact sequence

$$1 \longrightarrow C \xrightarrow{\;i\;} \widetilde{G} \xrightarrow{\;p\;} G \longrightarrow 1$$

such that $i(C)$ is contained in the center of $\widetilde{G}$. This is called a *central extension of G by C*.

1. For each $g \in G$, fix a representative $\widetilde{g} \in p^{-1}(g)$. Show that the failure of the map

$$g \mapsto \widetilde{g}$$

to be a group homomorphism is measured by a map

$$\beta : G \times G \longrightarrow C.$$

More precisely, show that there is a unique such β with

$$\widetilde{g} \cdot \widetilde{h} \;=\; i(\beta(g,h))\widetilde{gh}.$$

Proof. Since p is a group homomorphism, we have

$$p(\widetilde{g} \cdot \widetilde{h} \cdot (\widetilde{gh})^{-1}) = p(\widetilde{g}) \cdot p(\widetilde{h}) \cdot p(\widetilde{gh})^{-1} = g \cdot h \cdot (gh)^{-1} = 1.$$

In other words,

$$\widetilde{g} \cdot \widetilde{h} \cdot (\widetilde{gh})^{-1} \in ker(p) = im(i).$$

Since i is injective, there is a unique element of C mapping to $\widetilde{g} \cdot \widetilde{h} \cdot (\widetilde{gh})^{-1}$ under i. Call that element $\beta(g,h)$. □

2. Show that β satisfies the 2-cocycle condition

$$\beta(g,h)\beta(gh,k) \;=\; \beta(g,hk)\beta(h,k).$$

Proof. We will suppress i from the notation and treat $\beta(g,h)$ as an element of the central subgroup. Recalling that central elements commute with all elements, we obtain

$$\beta(g,h)\beta(gh,k) \cdot \widetilde{(gh)k} \;=\; \beta(g,h) \cdot \widetilde{gh} \cdot \widetilde{k} \;=\; (\widetilde{g} \cdot \widetilde{h}) \cdot \widetilde{k} \;=$$

$$\widetilde{g} \cdot (\widetilde{h} \cdot \widetilde{k}) \;=\; \widetilde{g} \cdot \beta(h,k) \cdot \widetilde{hk} \;=\; \beta(h,k) \cdot \widetilde{g} \cdot \widetilde{hk} \;=\; \beta(h,k)\beta(g,hk) \cdot \widetilde{g(hk)}.$$

Dividing both sides from the right by $\widetilde{ghk}$, we obtain the desired identity. □

3. Given a different choice of representatives with associated 2-cocycle β', show that there exists a map

$$\gamma : G \longrightarrow C$$

with

$$\gamma(g)\gamma(h)\beta(g,h) = \gamma(gh)\beta'(g,h).$$

Proof start. Denote the second choice of representatives by $g \mapsto \widetilde{\widetilde{g}}$. Throughout the argument, we supress i from the notation and view C as a subgroup of $\widetilde{G}$. Let

$$\gamma(g) := \widetilde{g} \cdot \widetilde{\widetilde{g}}^{-1}.$$

This is in C, because $p(\gamma(g)) = g \cdot g^{-1} = 1$. We have

$$\beta(g,h) \cdot \gamma(gh) \cdot \beta'(g,h)^{-1} = \beta(g,h) \cdot (\widetilde{gh}) \cdot \widetilde{\widetilde{gh}}^{-1} \cdot \beta'(g,h)^{-1} = \widetilde{g} \cdot \widetilde{h} \cdot \widetilde{\widetilde{h}}^{-1} \cdot \widetilde{\widetilde{g}}^{-1}$$

$$= \widetilde{g} \cdot \gamma(h) \cdot \widetilde{\widetilde{g}}^{-1} = \gamma(g) \cdot \gamma(h).$$

Here we have used the fact that elements of C are central in $\widetilde{G}$. ☐

4. Conversely, given a group G, and abelian group C and β as above, construct a central extension $\widetilde{G}_\beta$ of G by C. Given β' and γ, construct an isomorphism from $\widetilde{G}_\beta$ to $\widetilde{G}_{\beta'}$.

   Proof. The case that is normally considered is the scenario where we have the additional identities $\beta(1,g) = 1 = \beta(g,1)$ for all $g \in G$. We will limit ourselves to this special case. As a set

   $$\widetilde{G}_\beta := C \times G.$$

   The multiplication is defined as

   $$(c,g) \cdot (d,h) := (cd\beta(g,h), gh).$$

   The cocycle condition guarantees that this is associative, our additional identities ensure that $(1,1)$ is a neutral element. The inclusion i sends c to $(c,1)$, the quotient map p sends (c,g) to g. These are group homomorphisms. The isomorphism associated to γ sends (c,g) to $(c\gamma(g)^{-1}, g)$. ☐

5. Work through the examples of

   (a) the quaternion group

   $$1 \longrightarrow \{\pm 1\} \xrightarrow{\;i\;} Q \xrightarrow{\;p\;} K_4 \longrightarrow 1$$

   (b) the binary tetrahedral group

   $$1 \longrightarrow \{\pm 1\} \xrightarrow{\;i\;} 2T \xrightarrow{\;p\;} T \longrightarrow 1$$

   (c) the binary octahedral group

   $$1 \longrightarrow \{\pm 1\} \xrightarrow{\;i\;} 2O \xrightarrow{\;p\;} O \longrightarrow 1$$

(d) the binary icosahedral group

$$1 \longrightarrow \{\pm 1\} \xrightarrow{i} 2I \xrightarrow{p} I \longrightarrow 1.$$

Proof. The central subgroup of Q is $\{\pm 1\}$, the multiplication is determined by the table

$\cdot$	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

(first entry in row n times first entry in row m equals entry nm). Choose the set of representatives $\{1, i, j, k\}$ for, say $a = \{i, -i\}$ and $b := \{j, -j\}$ and $ab := \{k, -k\}$. Then β is represented by the table

β	1	a	b	ab
1	1	1	1	1
a	1	-1	1	-1
b	1	-1	-1	1
ab	1	1	-1	-1

You can look up the elements for the other groups on Wikipedia, choose representatives and write out the multiplication tables, if you wish. This is lengthy (a 60 times 60 table for 2I), but elementary. □

More on semi-direct products:

Find isomorphisms

$$2T \cong Q \rtimes (\mathbb{Z}/3\mathbb{Z})$$

and

$$O \cong T \rtimes (\mathbb{Z}/2\mathbb{Z}).$$

Finally, show that there is a short exact sequence

$$1 \longrightarrow Q \xrightarrow{i} 2O \xrightarrow{p} S_3 \longrightarrow 1$$

but no way to write the binary octahedral group as a semi-direct product of Q and S_3.

Proof. The first statement is an application of the third isomorphism theorem: Consider the sequence of inclusions

$$\{\pm 1\} < Q < 2T.$$

The quotient $Q/\{pm1\}$ is isomorphic to the Klein four group, the quotient $2T/\{\pm 1\}$ is the tetrahedral group T, and the inclusion is our old friend N from the first set of questions. Since N is normal in T, it follows that Q is normal in $2T$. Moreover, we have an isomorphism

$$2T/Q \cong T/N \cong \mathbb{Z}/3\mathbb{Z}.$$

Consider a rotation R of order 3 in T, and choose a representative $\widetilde{R}$ of $p^{-1}(R)$ in $2T$. Then $p(\widetilde{R}^3) = 1$, so $\widetilde{R}$ has either order 3 or order 6. In either case, $\widetilde{R}^2$ has order 3 and maps to R^2 under p. Hence the representatives for the cosets of Q inside $2T$ can be chosen as the elements of the cyclic subgroup $\langle \widetilde{R}^2 \rangle$ of order 3 in $2T$. If you look at the Wikipedia page on the binary tetrahedral group, you find a second description of the semi-direct product structure.

Now the second claim is the we can find a splitting for the short exact sequence

$$1 \longrightarrow A_4 \longrightarrow S_4 \xrightarrow{sgn} \{\pm 1\} \longrightarrow 1.$$

Note that any subgroup of index two is automatically normal. Such a splitting is given by any choice of odd permutation of order 2, for instance by (12). For the sake of completenes, we note that the conjugation action of (12) on N is as described above. The conjugation action on the remaining eight elements is

$$(123) \leftrightarrow (132) \qquad (124) \leftrightarrow (142) \qquad (134) \leftrightarrow (234) \qquad (143) \leftrightarrow (243).$$

Finally, the argument that Q is normal inside the binary octahedral group with quotient isomorphic to S_3 is another application of the third isomorphism theorem, very similar to the tetrahedral case. The only quaternion unit of order 2 is -1. In particular, there is no subgroup of $2O$ that is isomorphic to S_3. $\square$