Solutions for Tutorial 2

Topic 1: Algebras. 1. One definition of algebra is the following: Let R be a commutative ring. An R-algebra is a ring A, together with a ring homomorphism from R to A, such that the image of R is contained in the centre of A. Another is in terms of a scalar multiplication of R on A: an algebra is an R-module A, together with an R-bilinear ring multiplication, i.e., a binary operation $\cdot : A \times A \to A$ satisfying associativity and distributivity, possessing a unit 1 and such that

$$(\forall r \in R, a, b \in A) \qquad (ra) \cdot b = r(a \cdot b) = a \cdot (rb).$$

Indeed, given A satisfying the second definition, we have the map $r \mapsto r1_A$, which takes values in the centre of A. Conversely, any ring homomorphism $\eta : R \to A$ equips A with a scalar multiplication via $ra := \eta(r) \cdot a$, and if the image of η is central in A, the multiplication on A is R-bilinear.

2. The ring homomorphisms

$$\begin{aligned}
\mathbb{Z} &\to \mathbb{Z}/4\mathbb{Z} && (1)\\
a &\mapsto a + 4\mathbb{Z} &&\\
\mathbb{C} &\to Mat_{2\times 2}(\mathbb{C}) && (2)\\
z &\mapsto \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} &&\\
\mathbb{Z} &\hookrightarrow \mathbb{Z}[x] && (3)\\
\mathbb{Z} &\hookrightarrow \mathbb{Z}[i] && (4)\\
\mathbb{Z}[x] &\to \mathbb{Z}[i] && (5)\\
x &\mapsto i &&\\
\mathbb{Q} &\hookrightarrow \mathbb{Q}[i] && (6)\\
\mathbb{R} &\hookrightarrow \mathbb{H} && (7)
\end{aligned}$$

all have images contained in the centres of their targets. There are of course more possibilities, e.g.,

$$\begin{aligned}
\mathbb{Z}/24\mathbb{Z} &\to \mathbb{Z}/4\mathbb{Z}\\
a &\mapsto a + 4\mathbb{Z}.
\end{aligned}$$

Note that the inclusion of the complex numbers $\mathbb{C}$ in the quaternions $\mathbb{H}$ is not an example: $\mathbb{C}$ is not central in $\mathbb{H}$. The matrix example is maybe the most instructive, as you are used to thinking of the vector space of matrices. So, you may think of an algebra as a ring with some extra structure.

Topic 2: Quotient rings Let R be a ring, and let I be a two-sided ideal of R.

1. Show that there is a unique ring structure on the set of cosets R/I (with respect to which operation?) such that the quotient map

   $$q : R \longrightarrow R/I$$

   is a ring homomorphism. First, an ideal is a subgroup of $(R, +)$, which is an abelian group, so I is automatically normal. The monoid $(R, \cdot)$, on the other hand, might not be a group, and the ideal property $(\forall r \in R)\,(r \cdot I = I)$ illustrates just how much can go wrong when attempting to partition a monoid into "cosets". The quotient therefore refers to the cosets of I in R with respect to $+$, i.e.,

   $$\begin{aligned} q : R &\longrightarrow R/I \\ r &\longmapsto r + I. \end{aligned}$$

   This is surjective, and we already know that the quotient is a group with respect to $+$. Our only chance of defining a multiplication on R/I making q a ring map is to set $q(r) \cdot q(s) := q(r \cdot s)$. To see that this is well defined, assume $r \equiv r' \mod I$ and $s \equiv s' \mod I$. Then we have

   $$r \cdot s - r' \cdot s' \;=\; r(s - s') + (r - r')s' \;\in\; I.$$

   The last step uses that I is a two-sided ideal and is, in fact, the motivation for that definition.

2. The kernel of q is the coset of $0 + I = I$.

Topic 3: Third isomorphism theorem for rings Let I and J be two-sided ideals of R such that $I \subseteq J \subseteq R$.

1. Show that J/I is an ideal in the quotient ring R/I. First, J/I is non-empty, because it contains the coset I. Let a and b be elements of J, and let $r \in R$. Then we have $(a+I)-(b+I) = (a-b)+I$. This is again an element of J/I. Further $(r+I)\cdot(a+I) = r \cdot a + I$ is an element of J/I, and similarly for $a \cdot r$.
2. Show that there is a canonical isomorphism of rings

$$(R/I)/(J/I) \cong R/J.$$

By the first isomorphism theorem, it suffices to identify the kernel of the surjective map

$$\begin{array}{rcl} R/I & \longrightarrow & R/J \\ a+I & \longmapsto & a+J \end{array}$$

with the ideal J/I. Note first that this map is well defined, because $a \equiv b \mod I$ implies $a \equiv b \mod J$, i.e., the equivalence relation induced by J is stricktly coarser than that induced by I. The kernel consists of all cosets $a+I$ with $a+J = 0+J = J$. The latter is equivalent to $a \in J$.
3. Show that every ideal of R/I is of the form J/I for some ideal J of R containing I. Let $K \subseteq R/I$ be an ideal. Let $q : R \to R/I$ be the quotient map as in Topic 2, and set $J := q^{-1}(K)$. Then $k \in K$ if and only if $k = j + I$ for some $j \in J$. We need to show that J is an ideal. J is non-empty, since it contains 0. Given a and b in J and $r \in R$, we have $q(a-b) = q(a) - q(b) \in K$, so $a - b \in q^{-1}(K) = J$. Further $q(ra) = q(r)q(a)$ is in K, since $q(a)$ is, so ra is an element of J, and similarly for ar.