Reduced Expressions and Exchange Theorem

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Reference: C.Kassel and V.Turaev: Braid Groups section 4.4

Definition:

Given S_n, and i, j with $1 \leq i < j \leq n$, let $\tau_{i,j}$ be the transposition which excahnges i and j and fixes every other element in $\{1, ...n\}$. Denote the set of tranpositions in S_n by T.

Given $w \in S_n$, and inversion of w is a pair (i, j) with $1 \leq i < j \leq n$ and $w(i) > w(j)$. Denote by $I(w)$ the set $\{\tau_{i,j} \in T | (i, j)$ is an inversion of w$\}$. Note that $I(w)$ determines w uniquely.

Lemma:

Let $\tau_{i,j} \in T$ and $u \in S_n$. Then

$$u\tau_{i,j}u^{-1} = \begin{cases} \tau_{u(i),u(j)} & u(i) < u(j) \\ \tau_{u(j),u(i)} & u(i) > u(j) \end{cases}$$

Proof:

First note that $u\tau_{i,j}u^{-1}(u(j)) = u\tau_{i,j}(j) = u(i)$ and $u\tau_{i,j}u^{-1}(u(i)) = u\tau_{i,j}(i) = u(j)$. So $u\tau_{i,j}u^{-1}$ swaps $u(i)$ and $u(j)$. Further, if $1 \leq l \leq n$ has $l \neq u(i)$ and $l \neq u(j)$ then $l = u(l')$, $l' \neq i$ and $l' \neq j$. So $u\tau_{i,j}u^{-1}(l) = u\tau_{i,j}u^{-1}(u(l')) = u\tau_{i,j}(l') = u(l') = l$. So l is fixed. Thus

$$u\tau_{i,j}u^{-1} = \begin{cases} \tau_{u(i),u(j)} & u(i) < u(j) \\ \tau_{u(j),u(i)} & u(j) < u(i) \end{cases}$$

as claimed. □

Lemma:

Let $u, v \in S_n$. Then

$$I(uv) = v^{-1}I(u)v \triangle I(v)$$

Proof:

Let $\tau_{i,j} \in T$, $u, v \in S_n$. There are two cases. First suppose that $\tau_{i,j} \in I(v)$. We want to show that $\tau_{i,j} \notin v^{-1}I(u)v$ exactly when $\tau_{i,j} \in I(uv)$. We have that $\tau_{i,j} \in I(uv)$ iff $uv(j) < uv(i)$. Which is true in this case iff $(v(j), v(i))$ is not an inversion of u since $v(j) < v(i)$. Equivalently $v\tau_{i,j}v^{-1} = \tau_{v(j),v(i)} \notin I(u)$ and equivalently again $\tau_{i,j} \notin v^{-1}I(u)v$.

Now assume $\tau_{i,j} \notin I(v)$. We want to show that $\tau_{i,j} \in v^{-1}I(u)v$ exactly when $\tau_{i,j} \in I(uv)$. Then $\tau_{i,j} \in I(uv)$ iff $uv(i) > uv(j)$ which is true in this case iff $(v(i), v(j))$ is an inversion of u, since $v(i) < v(j)$. This is equivalent to $v\tau_{i,j}v^{-1} \in I(u)$ which is true exactly when $\tau_{i,j} \in v^{-1}I(u)v$.

We have now established that $\tau_{i,j} \in I(uv)$ iff $\tau_{i,j}$ is in exactly one of $I(v)$ or $v^{-1}I(u)v$. It is then clear that $I(uv) = v^{-1}I(u)v \triangle I(v)$. $\square$

Lemma:

Let $w \in S_n$, and let $s_{i_1}...s_{i_r}$ be a reduced expression for w. Define the set of transpositions $\{t_1, ..., t_r\} \subseteq T$ where $t_k = s_{i_r}...s_{i_{k+1}}s_{i_k}s_{i_{k+1}}...s_{i_r}$. Then:
(i) $\forall k, wt_k = s_{i_1}...\hat{s_{i_k}}...s_{i_r}$ where the hat indicates that s_k has been removed.
(ii) $t_l \neq t_k$ for $k \neq l$
(iii) $I(w) = \{t_1, ..., t_r\}$

Proof:

(i) Let $w \in S_n$ and let $s_{i_1}...s_{i_r}$ be a reduced expression for w. Given t_k, $1 \leq k \leq r$, we have

$$\begin{aligned} wt_k &= (s_{i_1}...s_{i_r})(s_{i_r}...s_{i_{k+1}}s_{i_k}s_{i_{k+1}}...s_{i_r}) \\ &= (s_{i_1}...s_{i_{k-1}})(s_{i_k}...s_{i_r})(s_{i_r}...s_{i_k})(s_{i_{k+1}}...s_{i_r}) \\ &= s_{i_1}...\hat{s_{i_k}}...s_{i_r} \end{aligned}$$

(ii) It is clear then that all t_i are distinct, since if $t_l = t_k$, $l \neq k$ we can assume $l < k$. Then

$$w = wt_l^2 = wt_lt_k = s_{i_1}...\hat{s_{i_l}}...\hat{s_{i_k}}...s_{i_r}$$

Contradicting that $s_{i_1}...s_{i_r}$ is a reduced expression for w.

(iii) We will do induction on the string $s_{i_1}...s_{i_r}$. It is clear in the base case that $I(s_{i_r}) = \{s_{i_r}\} = \{t_r\}$. Now assume that for some $1 < k < r$ we have

$I(s_{i_k}...s_{i_r}) = \{t_k, ..., t_r\}$. Then

$$\begin{aligned} I(s_{i_{k-1}}s_{i_k}...s_{i_r}) &= (s_{i_k}...s_{i_r})^{-1}I(s_{i_{k-1}})(s_{i_k}...s_{i_r})\triangle I(s_{i_k}...s_{i_r}) \\ &= (s_{i_k}...s_{i_r})^{-1}\{s_{i_k}\}(s_{i_k}...s_{i_r}\}\triangle\{t_k, ..., t_r\} \\ &= \{t_{k-1}\}\triangle\{t_k, ..., t_r\} = \{t_{k-1}\} \cup \{t_k, ..., t_r\} \\ &= \{t_{k-1}, ..., t_r\} \end{aligned}$$

It follows by induction that $I(w) = I(s_{i_1}...s_{i_r}) = \{t_1, ..., t_r\}$ □

Corollary:

For $w \in S_n$, then we have $\lambda(w) = |I(w)|$. Also given a reduced expression $s_{i_1}...s_{i_r}$ for w, and $\tau \in I(w)$, then $w\tau = s_{i_1}...\hat{s_{i_k}}...s_{i_r}$ for some $1 \le k \le r$.

Proof:

This is clear since given $w \in S_n$ of length r, we have a reduced expression $s_{i_1}...s_{i_r}$ and $I(w) = \{t_1, ..., t_r\}$ where t_i is defined as in the above lemma. Further $|I(w)| = |\{t_1, ..., t_r\}| = r$. If $\tau \in I(w)$, then $\tau = t_k$ for some k, and by the calculation above $w\tau = s_{i_1}...\hat{s_{i_k}}...s_{i_r}$. □

Lemma:

For $\tau \in T$, $\lambda(w\tau) \neq \lambda(w)$ and $\tau \in I(w)$ if and only if $\lambda(w\tau) < \lambda(w)$.

Proof:

In the above lemma we saw that for $\tau \in I(w)$, $w\tau = s_{i_1}...\hat{s_{i_k}}...s_{i_r}$ for some $1 \le k \le r$ where $s_{i_1}...s_{i_r}$ is a reduced expression for w. It follows that $\lambda(w) > \lambda(w\tau)$. Conversely if $\tau \notin I(w)$, then $\tau \notin \tau I(w)\tau$. So, $\tau \in \tau I(w)\tau\triangle\{\tau\} = I(w\tau)$. It follows then that $\lambda(w) = \lambda(w\tau^2) < \lambda(w\tau)$. □

Lemma:

for $w \in S_n$, $\lambda(ws_i) = \lambda(w) - 1$ if and only if $w(i) > w(i+1)$.

Proof:

From two previous lemmas, we have that $\lambda(ws_i) = \lambda(w) \pm 1$ and $\lambda(ws_i) < \lambda(w)$ exactly when $(i, i+1)$ is an inversion of w. The claim follows.

Theorem: (Exchange Theorem)

Let $s_{i_1}...s_{i_r}$ be a reduced expression for $w \in S_n$. Then, given a simple transposition s_j, If $\lambda(ws_j) < \lambda(w)$ then there exists a $1 \leq k \leq r$ such that $ws_j = s_{i_1}...\hat{s_{i_k}}...s_{i_r}$. Similarly if $\lambda(s_j w) < \lambda(w)$, there is a $1 \leq k \leq r$ such that $s_j w = s_{i_1}...\hat{s_{i_k}}...s_{i_r}$.

Proof:

Let $w \in S_n$ have a reduced expression $s_{i_1}...s_{i_r}$. If $\lambda(ws_j) < \lambda(s_j)$, then $s_j \in I(w)$ so $s_j) = s_{i_1}...\hat{s_{i_k}}...s_{i_r}$, by the above corollary, so we have the first part. For the second part, we have that if $\lambda(s_j w) < \lambda(w)$, then $\lambda(w^{-1}s_j) = \lambda(s_j w) < \lambda(w) = \lambda(w^{-1})$, so $s_j \in I(w^{-1})$. Taking the reduced expression for w^{-1}, $s_{i_r}...s_{i_1}$ we have that $w^{-1}s_j = s_{i_r}...\hat{s_{i_k}}...s_{i_1}$ and further that $s_j w = s_{i_1}...\hat{s_{i_k}}...s_{i_r}$ $\square$

Corollary:

Let $w \in S_n$. $\lambda(ws_j) < \lambda(w)$ iff there is a reduced expression for w beginning with s_j.

Proof:

If $\lambda(ws_j) < \lambda(w)$, we have $ws_j = s_{i_1}...\hat{s_{i_k}}...s_{i_r}$ wheres $s_{i_1}...s_{i_r}$ is reduced expression for w. Then however $w = s_{i_1}...\hat{s_{i_k}}...s_{i_r}s_j$ which is a reduced expression for w. The converse is obvious.

Lemma:

If $w \in S_n$ and simple transpositions $s_i, s_j \in S_n$, if $\lambda(s_i w s_j) = \lambda(w)$ and $\lambda(s_i w) = \lambda(ws_j)$ then $s_i w = ws_j$.

Proof:

We can't have $\lambda(w) = \lambda(ws_j)$ so we have two cases. If $\lambda(s_i w) = \lambda(ws_j) > \lambda(s_i w s_j) = \lambda(w)$.

$$I(s_i w) = \{w^{-1}s_i w\} \triangle I(w)$$

$\lambda(ws_j) > \lambda(w)$, so $s_j \notin I(w)$. However, $\lambda(s_i w s_j) < \lambda(s_i w)$, so $s_j \in I(s_i w)$. It follows that $s_j = w^{-1}s_i w$ and the claim is clear.

Alternatively, assume that $\lambda(s_i w) < \lambda(w)$. We have

$$\begin{aligned} I(w) &= I(s_i^2 w) \\ &= w^{-1}s_i\{s_i\}s_i w \triangle I(s_i w) \\ &= \{w^{-1}s_i w\} \triangle I(s_i w) \end{aligned}$$

$s_j \notin I(s_i w)$ but $s_j \in I(w)$. It follows that $s_j = w^{-1}s_i w$ and we are done. $\square$