Tutorial 6 – Algebra 2019

Let $F \subset K$ be fields, and let a be an element of K.

(1) Recall what it means for a to be algebraic over F.

(2) In class, you saw a fast forward version of the proof that the field extension generated by F and a is isomorphic to the quotient of a polynomial algebra. In this tutorial, you will fill in the details.

  (a) Recall the definition of the relevant map ϕ from a polynomial algebra to K.
  (b) Show that $im(\phi)$ is an integral domain.
  (c) Using the first isomorphism theorem, argue that $ker(\phi)$ is a prime ideal.
  (d) Show that $F[x]$ is a PID.
  (e) Prove that $ker(phi)$ is maximal.
  (f) Prove that $ker(phi) = (p(x))$ where $p(x)$ is an irreducible polynomial.
  (g) Prove that $im(\phi)$ is a field.
  (h) Prove $im(\phi) = F(a)$.
  (i) Describe $F(a)$ as a quotient of the polynomial algebra $F[x]$.

(3) Work through some examples. A good place to get a feel for what is going on is the extension $\mathbb{F}_2 \subset \mathbb{F}_{16}$.