Assignment 1 - Commutative and Multilinear Algebra

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Show that the symmetric and the exterior powers are exponential in the following sense. There exist natural isomorphisms.

a)
$$S^n(V \oplus W) \cong \bigoplus_{k+l=n} S^k(V) \otimes S^l(W)$$

b)
$$\Lambda^n(V \oplus W) \cong \bigoplus_{k+l=n} \Lambda^k(V) \otimes \Lambda^l(W)$$

Here V and W are vector spaces over a fixed field, $\mathbf{k}$ and the tensor product and direct sum are taken over $\mathbf{k}$.

a) We wish to that that there exists a natural transformation between the functors $S^n(- \oplus -)$ and $\bigoplus_{k+l=n} S^k(-) \otimes S^l(-)$.

Let $\mathbf{k}$ be a field and V, W, X, Y be vector spaces over $\mathbf{k}$ with linear maps $\alpha : V \to X$ and $\beta : W \to Y$.

This means we need to find a family of isomorphisms $\varphi_{V,W}$ and $\varphi_{X,Y}$ that make the following diagrams commute for all $V, W, X, Y, \alpha, \beta$.

$$\begin{array}{ccc} S^n(V \oplus W) & \overset{\varphi_{V,W}}{\to} & \bigoplus_{k+l=n} S^k(V) \otimes S^l(W) \\ \downarrow_{S^n(\alpha \oplus \beta)} & & \downarrow_{\bigoplus_{k+l=n} S^k(\alpha) \otimes S^l(\beta)} \\ S^n(X \oplus Y) & \overset{\varphi_{X,Y}}{\to} & \bigoplus_{k+l=n} S^k(X) \otimes S^l(Y) \end{array}$$

Let C, D, E, F be ordered sets such that $\{v_c\}_{c \in C}$ is an ordered basis for V, $\{w_d\}_{d \in D}$ is an ordered basis for W, $\{x_e\}_{e \in E}$ is an ordered basis for X, $\{y_f\}_{f \in F}$ is an ordered basis for Y.

We will abuse notation slightly and consider $v_c, w_d \in V \oplus W$ and $x_e, y_f \in X \oplus Y$. Then $\{\{v_c\}_{c\in C}, \{w_d\}_{d\in D}\}$ is an ordered basis for $V \oplus W$ and $\{\{x_e\}_{e\in E}, \{y_f\}_{f\in F}\}$ is a basis for $X \oplus Y$.

This means that

$$v_{i_1} \cdot ... \cdot v_{i_k} \cdot w_{j_1} \cdot ... \cdot w_{j_l}$$

with $k + l = n$, $i_1, ..., i_k \in C$, $j_1, ..., j_l \in D$ with $i_1 \leq ... \leq i_k$ and $j_1 \leq ... \leq j_l$ forms a basis for $S^n(V \oplus W)$. Similarly

$$x_{i_1} \cdot ... \cdot x_{i_k} \cdot y_{j_1} \cdot ... \cdot y_{j_l}$$

with $k + l = n$, $i_1, ..., i_k \in E$, $j_1, ..., j_l \in F$ with $i_1 \leq ... \leq i_k$ and $j_1 \leq ... \leq j_l$ forms a basis for $S^n(X \oplus Y)$.

Now we also have

$$(v_{i_1} \cdot ... \cdot v_{i_k}) \otimes (w_{j_1} \cdot ... \cdot w_{j_l})$$

with $k + l = n$, $i_1, ..., i_k \in C$, $j_1, ..., j_l \in D$ with $i_1 \leq ... \leq i_k$ and $j_1 \leq ... \leq j_l$ forms a basis for $\bigoplus_{k+l=n} S^k(V) \otimes S^l(W)$. Similarly

$$(x_{i_1} \cdot ... \cdot x_{i_k}) \otimes (y_{j_1} \cdot ... \cdot y_{j_l})$$

with $k + l = n$, $i_1, ..., i_k \in E$, $j_1, ..., j_l \in F$ with $i_1 \leq ... \leq i_k$ and $j_1 \leq ... \leq j_l$ forms a basis for $\bigoplus_{k+l=n} S^k(X) \otimes S^l(Y)$.

Define the linear map on the basis vectors as follows

$$\varphi_{V,W} : S^n(V \oplus W) \to \bigoplus_{k+l=n} S^k(V) \otimes S^l(W) \quad s.t$$

$$\varphi_{V,W}(v_{i_1} \cdot ... \cdot v_{i_k} \cdot w_{j_1} \cdot ... \cdot w_{j_l}) = (v_{i_1} \cdot ... \cdot v_{i_k}) \otimes (w_{j_1} \cdot ... \cdot w_{j_l})$$

$$\varphi_{X,Y} : S^n(X \oplus Y) \to \bigoplus_{k+l=n} S^k(X) \otimes S^l(Y) \quad s.t$$

$$\varphi_{X,Y}(x_{i_1} \cdot ... \cdot x_{i_k} \cdot y_{j_1} \cdot ... \cdot y_{j_l}) = (x_{i_1} \cdot ... \cdot x_{i_k}) \otimes (y_{j_1} \cdot ... \cdot y_{j_l})$$

We can see that these maps are bijections between the basis vectors and so are isomorphisms of vector spaces. We now wish to show that these isomorphisms doesn't depend on the choice of bases.

Let $\{r_c\}_{c\in C}$ be another ordered basis for V and $\{s_d\}_{d\in D}$ another ordered basis for W. We will abuse notation slightly and consider $r_i, s_j \in V \oplus W$. Then $\{\{r_c\}_{c\in C}, \{s_d\}_{d\in D}\}$ is an ordered basis for $V \oplus W$.

This means that

$$r_{i_1} \cdot ... \cdot r_{i_k} \cdot s_{j_1} \cdot ... \cdot s_{j_l}$$

with $k+l=n$, $i_1,...,i_k \in C$, $j_1,...,j_l \in D$ with $i_1 \leq ... \leq i_k$ and $j_1 \leq ... \leq j_l$ forms a basis for $S^n(V \oplus W)$.

Now we also have

$$(r_{i_1} \cdot ... \cdot r_{i_k}) \otimes (s_{j_1} \cdot ... \cdot s_{j_l})$$

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Define the linear map on the basis vectors as follows

$$\phi_{V,W} : S^n(V \oplus W) \to \bigoplus_{k+l=n} S^k(V) \otimes S^l(W) \quad s.t$$

$$\phi_{V,W}(r_{i_1} \cdot ... \cdot r_{i_k} \cdot s_{j_1} \cdot ... \cdot s_{j_l}) = (r_{i_1} \cdot ... \cdot r_{i_k}) \otimes (s_{j_1} \cdot ... \cdot s_{j_l})$$

We can see that this map is a bijection between the basis vectors and so is an isomorphism of vector spaces. We now wish to show that this isomorphism is the same as our initial isomorphism $\varphi_{V,W}$.

We wish to show $\varphi_{V,W} = \phi_{V,W}$ and $\varphi_{X,Y} = \phi_{X,Y}$.

Let $A \in GL(V), B \in GL(W)$ be the change of basis matrices such that $r_i = \sum_{j \in C} A_{ij} v_j$, $s_i = \sum_{j \in D} B_{ij} w_j$.

Consider the difference between the images of the basis vectors.

$$\varphi_{V,W}(r_{i_1} \cdot ... \cdot r_{i_k} \cdot s_{j_1} \cdot ... \cdot s_{j_l}) - \phi_{V,W}(r_{i_1} \cdot ... \cdot r_{i_k} \cdot s_{j_1} \cdot ... \cdot s_{j_l})$$

$$= \varphi_{V,W}\left(\left(\sum_{m_1 \in C} A_{i_1 m_1} v_{m_1}\right) \cdot ... \cdot \left(\sum_{m_k \in C} A_{i_k m_k} v_{m_k}\right) \cdot \left(\sum_{n_1 \in D} B_{j_1 n_1} w_{n_1}\right) \cdot ... \cdot \left(\sum_{n_l \in D} B_{j_l n_l} w_{n_l}\right)\right)$$
$$-(r_{i_1} \cdot ... \cdot r_{i_k}) \otimes (s_{j_1} \cdot ... \cdot s_{j_l})$$

$$= \sum_{m_1,...,m_k \in C} \sum_{n_1,...,n_l \in D} A_{i_1 m_1} ... A_{i_k m_k} B_{j_1 n_1} ... B_{j_l n_l} \varphi_{V,W}\left(v_{m_1} \cdot ... \cdot v_{m_k} \cdot w_{m_1} \cdot ... \cdot w_{m_l}\right)$$
$$-(r_{i_1} \cdot ... \cdot r_{i_k}) \otimes (s_{j_1} \cdot ... \cdot s_{j_l})$$

$$= \sum_{m_1,...,m_k \in C} \sum_{n_1,...,n_l \in D} A_{i_1 m_1} ... A_{i_k m_k} B_{j_1 n_1} ... B_{j_l n_l} (v_{m_1} \cdot ... \cdot v_{m_k} \otimes w_{m_1} \cdot ... \cdot w_{m_l})$$
$$-(r_{i_1} \cdot ... \cdot r_{i_k}) \otimes (s_{j_1} \cdot ... \cdot s_{j_l})$$

$$= \left(\left(\sum_{m_1 \in C} A_{i_1 m_1} v_{m_1}\right) \cdot ... \cdot \left(\sum_{m_k \in C} A_{i_k m_k} v_{m_k}\right) \otimes \left(\sum_{n_1 \in D} B_{j_1 n_1} w_{n_1}\right) \cdot ... \cdot \left(\sum_{n_l \in D} B_{j_l n_l} w_{n_l}\right)\right)$$
$$-(r_{i_1} \cdot ... \cdot r_{i_k}) \otimes (s_{j_1} \cdot ... \cdot s_{j_l})$$

$$= (r_{i_1} \cdot ... \cdot r_{i_k}) \otimes (s_{j_1} \cdot ... \cdot s_{j_l}) - (r_{i_1} \cdot ... \cdot r_{i_k}) \otimes (s_{j_1} \cdot ... \cdot s_{j_l}) = 0$$

So for any basis vector $r_{i_1} \cdot ... \cdot r_{i_k} \cdot s_{j_1} \cdot ... \cdot s_{j_l}$ we have

$$\varphi_{V,W}(r_{i_1} \cdot ... \cdot r_{i_k} \cdot s_{j_1} \cdot ... \cdot s_{j_l}) = \phi_{V,W}(r_{i_1} \cdot ... \cdot r_{i_k} \cdot s_{j_1} \cdot ... \cdot s_{j_l})$$

This shows that $\varphi_{V,W} = \phi_{V,W}$ as they are both defined to be linear maps and the images of the basis vectors to the same for each map.

Similarly it can be shown that $\varphi_{X,Y}$ does not depend on the initial choice of bases x_i and y_j.

This means that we can define $\varphi_{V,W}$ for any V, W without dependence on a choice of a basis for V, W. So is a candidate for a natural transformation between the functors $S^n(- \oplus -)$ and $\bigoplus_{k+l=n} S^k(-) \otimes S^l(-)$.

To show that $\varphi_{V,W}$ and $\varphi_{X,Y}$ are natural we now need to consider the following compositions

$$\varphi_{X,Y} \circ (S^n(\alpha \oplus \beta))$$

$$\left(\bigoplus_{k+l=n} S^k(\alpha) \otimes S^l(\beta) \right) \circ \varphi_{V,W}$$

Now let $\alpha(v_i) = \sum_{j \in E} \alpha_{ij} x_j$ and $\beta(w_i) = \sum_{j \in F} \beta_{ij} y_j$.

$$\varphi_{X,Y} \circ (S^n(\alpha \oplus \beta)) (v_{i_1} \cdot ... \cdot v_{i_k} \cdot w_{j_1} \cdot ... \cdot w_{j_l})$$

$$= \varphi_{X,Y}(\alpha(v_{i_1}) \cdot ... \cdot \alpha(v_{i_k}) \cdot \beta(w_{j_1}) \cdot ... \cdot \beta(j_l))$$

$$= \varphi_{X,Y} \left(\left(\sum_{k_1 \in E} \alpha_{i_1 k_1} x_{k_1} \right) \cdot ... \cdot \left(\sum_{k_k \in E} \alpha_{i_k k_k} x_{k_k} \right) \cdot \left(\sum_{l_1 \in F} \alpha_{j_1 l_1} x_{l_1} \right) \cdot ... \cdot \left(\sum_{l_l \in F} \alpha_{j_l l_l} x_{l_l} \right) \right)$$

$$= \varphi_{X,Y} \left(\sum_{k_1,...,k_k \in E} \sum_{l_1,...,l_l \in F} \alpha_{i_1,k_1} ... \alpha_{i_k k_k} \beta_{j_1 l_1} ... \beta_{j_l l_l} (x_{k_1} \cdot ... \cdot x_{k_k} \cdot y_{l_1} \cdot ... \cdot y_{l_l}) \right)$$

$$= \sum_{k_1,...,k_k \in E} \sum_{l_1,...,l_l \in F} \alpha_{i_1,k_1} ... \alpha_{i_k k_k} \beta_{j_1 l_1} ... \beta_{j_l l_l} (x_{k_1} \cdot ... \cdot x_{k_k} \otimes y_{l_1} \cdot ... \cdot y_{l_l})$$

$$= \left(\left(\sum_{k_1 \in E} \alpha_{i_1 k_1} x_{k_1} \right) \cdot ... \cdot \left(\sum_{k_k \in E} \alpha_{i_k k_k} x_{k_k} \right) \right) \otimes \left(\left(\sum_{l_1 \in F} \alpha_{j_1 l_1} x_{l_1} \right) \cdot ... \cdot \left(\sum_{l_l \in F} \alpha_{j_l l_l} x_{l_l} \right) \right)$$

$$= (\alpha(v_{i_1}) \cdot ... \cdot \alpha(v_{i_k})) \otimes (\beta(w_{j_1}) \cdot ... \cdot \beta(j_l))$$

$$= \left(\bigoplus_{k+l=n} S^k(\alpha) \otimes S^l(\beta) \right) ((v_{i_1} \cdot ... \cdot v_{i_k}) \otimes (w_{j_1} \cdot ... \cdot w_{j_l}))$$

$$= \left(\bigoplus_{k+l=n} S^k(\alpha) \otimes S^l(\beta) \right) \circ \varphi_{V,W}(v_{i_1} \cdot ... \cdot v_{i_k} \cdot w_{j_1} \cdot ... \cdot w_{j_l})$$

Now we have two linear maps that agree on each basis vector so

$$\varphi_{X,Y} \circ (S^n(\alpha \oplus \beta)) = \left(\bigoplus_{k+l=n} S^k(\alpha) \otimes S^l(\beta) \right) \circ \varphi_{V,W}$$

This shows that $\varphi_{(-,-)}$ is a natural transformation between the functors $S^n(- \oplus -)$ and $\bigoplus_{k+l=n} S^k(-) \otimes S^l(-)$.

b) We wish to that that there exists a natural transformation between the functors $\Lambda^n(- \oplus -)$ and $\bigoplus_{k+l=n} \Lambda^k(-) \otimes \Lambda^l(-)$.

Let $\mathbf{k}$ be a field and V, W, X, Y be vector spaces over $\mathbf{k}$ with linear maps $\alpha : V \to X$ and $\beta : W \to Y$.

This means we need to find a family of isomorphisms $\varphi_{V,W}$ and $\varphi_{X,Y}$ that make the following diagrams commute for all $V, W, X, Y, \alpha, \beta$.

$$\begin{array}{ccc} \Lambda^n(V \oplus W) & \xrightarrow{\varphi_{V,W}} & \bigoplus_{k+l=n} \Lambda^k(V) \otimes \Lambda^l(W) \\ \downarrow {\scriptstyle \Lambda^n(\alpha \oplus \beta)} & & \downarrow {\scriptstyle \oplus_{k+l=n} \Lambda^k(\alpha) \otimes \Lambda^l(\beta)} \\ \Lambda^n(X \oplus Y) & \xrightarrow{\varphi_{X,Y}} & \bigoplus_{k+l=n} \Lambda^k(X) \otimes \Lambda^l(Y) \end{array}$$

Let C, D, E, F be ordered sets such that $\{v_c\}_{c \in C}$ is an ordered basis for V, $\{w_d\}_{d \in D}$ is an ordered basis for W, $\{x_e\}_{e \in E}$ is an ordered basis for X, $\{y_f\}_{f \in F}$ is an ordered basis for Y. We will abuse notation slightly and consider $v_c, w_d \in V \oplus W$ and $x_e, y_f \in X \oplus Y$. Then $\{\{v_c\}_{c \in C}, \{w_d\}_{d \in D}\}$ is an ordered basis for $V \oplus W$ and $\{\{x_e\}_{e \in E}, \{y_f\}_{f \in F}\}$ is a basis for $X \oplus Y$.

This means that

$$v_{i_1} \wedge ... \wedge v_{i_k} \wedge w_{j_1} \wedge ... \wedge w_{j_l}$$

with $k + l = n$, $i_1, ..., i_k \in C$, $j_1, ..., j_l \in D$ with $i_1 < ... < i_k$ and $j_1 < ... < j_l$ forms a basis for $\Lambda^n(V \oplus W)$. Similarly

$$x_{i_1} \wedge ... \wedge x_{i_k} \wedge y_{j_1} \wedge ... \wedge y_{j_l}$$

with $k + l = n$, $i_1, ..., i_k \in E$, $j_1, ..., j_l \in F$ with $i_1 < ... < i_k$ and $j_1 < ... < j_l$ forms a basis for $\Lambda^n(X \oplus Y)$.

Now we also have

$$(v_{i_1} \wedge ... \wedge v_{i_k}) \otimes (w_{j_1} \wedge ... \wedge w_{j_l})$$

with $k+l=n$, $i_1,...,i_k \in C$, $j_1,...,j_l \in D$ with $i_1 < ... < i_k$ and $j_1 < ... < j_l$ forms a basis for $\bigoplus_{k+l=n} \Lambda^k(V) \otimes \Lambda^l(W)$. Similarly

$$(x_{i_1} \wedge ... \wedge x_{i_k}) \otimes (y_{j_1} \wedge ... \wedge y_{j_l})$$

with $k+l=n$, $i_1,...,i_k \in E$, $j_1,...,j_l \in F$ with $i_1 < ... < i_k$ and $j_1 < ... < j_l$ forms a basis for $\bigoplus_{k+l=n} \Lambda^k(X) \otimes \Lambda^l(Y)$.

Define the linear map on the basis vectors as follows

$$\varphi_{V,W} : \Lambda^n(V \oplus W) \to \bigoplus_{k+l=n} \Lambda^k(V) \otimes \Lambda^l(W) \quad s.t$$

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We can see that these maps are bijections between the basis vectors and so are isomorphisms of vector spaces. We now wish to show that these isomorphisms doesn't depend on the choice of bases.

Let $\{r_c\}_{c \in C}$ be another ordered basis for V and $\{s_d\}_{d \in D}$ another ordered basis for W. We will abuse notation slightly and consider $r_i, s_j \in V \oplus W$. Then $\{\{r_c\}_{c \in C}, \{s_d\}_{d \in D}\}$ is an ordered basis for $V \oplus W$.

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We can see that this map is a bijection between the basis vectors and so is an isomorphism of vector spaces. We now wish to show that this isomorphism is the same as our initial isomorphism $\varphi_{V,W}$.

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Consider the difference between the images of the basis vectors.

$$\varphi_{V,W}(r_{i_1} \wedge ... \wedge r_{i_k} \wedge s_{j_1} \wedge ... \wedge s_{j_l}) - \phi_{V,W}(r_{i_1} \wedge ... \wedge r_{i_k} \wedge s_{j_1} \wedge ... \wedge s_{j_l})$$

$$= \varphi_{V,W}\left(\left(\sum_{m_1\in C} A_{i_1m_1}v_{m_1}\right) \wedge ... \wedge \left(\sum_{m_k\in C} A_{i_km_k}v_{m_k}\right) \wedge \left(\sum_{n_1\in D} B_{j_1n_1}w_{n_1}\right) \wedge ... \wedge \left(\sum_{n_l\in D} B_{j_ln_l}w_{n_l}\right)\right)$$

$$-(r_{i_1} \wedge ... \wedge r_{i_k}) \otimes (s_{j_1} \wedge ... \wedge s_{j_l})$$

$$= \sum_{m_1,...,m_k\in C} \sum_{n_1,...,n_l\in D} A_{i_1m_1}...A_{i_km_k}B_{j_1n_1}...B_{j_ln_l}\varphi_{V,W}\left(v_{m_1} \wedge ... \wedge v_{m_k} \wedge w_{m_1} \wedge ... \wedge w_{m_l}\right)$$

$$-(r_{i_1} \wedge ... \wedge r_{i_k}) \otimes (s_{j_1} \wedge ... \wedge s_{j_l})$$

$$= \sum_{m_1,...,m_k\in C} \sum_{n_1,...,n_l\in D} A_{i_1m_1}...A_{i_km_k}B_{j_1n_1}...B_{j_ln_l}(v_{m_1} \wedge ... \wedge v_{m_k} \otimes w_{m_1} \wedge ... \wedge w_{m_l})$$

$$-(r_{i_1} \wedge ... \wedge r_{i_k}) \otimes (s_{j_1} \wedge ... \wedge s_{j_l})$$

$$= \left(\left(\sum_{m_1\in C} A_{i_1m_1}v_{m_1}\right) \wedge ... \wedge \left(\sum_{m_k\in C} A_{i_km_k}v_{m_k}\right) \otimes \left(\sum_{n_1\in D} B_{j_1n_1}w_{n_1}\right) \wedge ... \wedge \left(\sum_{n_l\in D} B_{j_ln_l}w_{n_l}\right)\right)$$

$$-(r_{i_1} \wedge ... \wedge r_{i_k}) \otimes (s_{j_1} \wedge ... \wedge s_{j_l})$$

$$= (r_{i_1} \wedge ... \wedge r_{i_k}) \otimes (s_{j_1} \wedge ... \wedge s_{j_l}) - (r_{i_1} \wedge ... \wedge r_{i_k}) \otimes (s_{j_1} \wedge ... \wedge s_{j_l}) = 0$$

So for any basis vector $r_{i_1} \wedge ... \wedge r_{i_k} \wedge s_{j_1} \wedge ... \wedge s_{j_l}$ we have

$$\varphi_{V,W}(r_{i_1} \wedge ... \wedge r_{i_k} \wedge s_{j_1} \wedge ... \wedge s_{j_l}) = \phi_{V,W}(r_{i_1} \wedge ... \wedge r_{i_k} \wedge s_{j_1} \wedge ... \wedge s_{j_l})$$

This shows that $\varphi_{V,W} = \phi_{V,W}$ as they are both defined linear maps and the images of the basis vectors to the same for each map.

Similarly it can be shown that $\varphi_{X,Y}$ does not depend on the initial choice of bases x_i and y_j.

This means that we can define $\varphi_{V,W}$ for any V, W without dependence on the choice of a basis for V, W. So is a candidate for a natural transformation between the functors $\Lambda^n(- \oplus -)$ and $\bigoplus_{k+l=n} \Lambda^k(-) \otimes \Lambda^l(-)$.

To show that $\varphi_{V,W}$ and $\varphi_{X,Y}$ are natural we now need to consider the following compositions

$$\varphi_{X,Y} \circ (\Lambda^n(\alpha \oplus \beta))$$

$$\left(\bigoplus_{k+l=n} \Lambda^k(\alpha) \otimes \Lambda^l(\beta)\right) \circ \varphi_{V,W}$$

Now let $\alpha(v_i) = \sum_{j\in E} \alpha_{ij} x_j$ and $\beta(w_i) = \sum_{j\in F} \beta_{ij} y_j$.

$$\varphi_{X,Y} \circ (\Lambda^n(\alpha \oplus \beta))\,(v_{i_1} \wedge ... \wedge v_{i_k} \wedge w_{j_1} \wedge ... \wedge w_{j_l})$$

$$= \varphi_{X,Y}(\alpha(v_{i_1}) \wedge ... \wedge \alpha(v_{i_k}) \wedge \beta(w_{j_1}) \wedge ... \wedge \beta(j_l))$$

$$= \varphi_{X,Y}\left(\left(\sum_{k_1\in E} \alpha_{i_1 k_1} x_{k_1}\right) \wedge ... \wedge \left(\sum_{k_k\in E} \alpha_{i_k k_k} x_{k_k}\right) \wedge \left(\sum_{l_1\in F} \alpha_{j_1 l_1} x_{l_1}\right) \wedge ... \wedge \left(\sum_{l_l\in F} \alpha_{j_l l_l} x_{l_l}\right)\right)$$

$$= \varphi_{X,Y}\left(\sum_{k_1,...,k_k\in E}\ \sum_{l_1,...,l_l\in F} \alpha_{i_1,k_1}...\alpha_{i_k k_k}\beta_{j_1 l_1}...\beta_{j_l l_l}(x_{k_1} \wedge ... \wedge x_{k_k} \wedge y_{l_1} \wedge ... \wedge y_{l_l})\right)$$

$$= \sum_{k_1,...,k_k\in E}\ \sum_{l_1,...,l_l\in F} \alpha_{i_1,k_1}...\alpha_{i_k k_k}\beta_{j_1 l_1}...\beta_{j_l l_l}(x_{k_1} \wedge ... \wedge x_{k_k} \otimes y_{l_1} \wedge ... \wedge y_{l_l})$$

$$= \left(\left(\sum_{k_1\in E} \alpha_{i_1 k_1} x_{k_1}\right) \wedge ... \wedge \left(\sum_{k_k\in E} \alpha_{i_k k_k} x_{k_k}\right)\right) \otimes \left(\left(\sum_{l_1\in F} \alpha_{j_1 l_1} x_{l_1}\right) \wedge ... \wedge \left(\sum_{l_l\in F} \alpha_{j_l l_l} x_{l_l}\right)\right)$$

$$= (\alpha(v_{i_1}) \wedge ... \wedge \alpha(v_{i_k})) \otimes (\beta(w_{j_1}) \wedge ... \wedge \beta(j_l))$$

$$= \left(\bigoplus_{k+l=n} \Lambda^k(\alpha) \otimes \Lambda^l(\beta)\right)((v_{i_1} \wedge ... \wedge v_{i_k}) \otimes (w_{j_1} \wedge ... \wedge w_{j_l}))$$

$$= \left(\bigoplus_{k+l=n} \Lambda^k(\alpha) \otimes \Lambda^l(\beta)\right) \circ \varphi_{V,W}(v_{i_1} \wedge ... \wedge v_{i_k} \wedge w_{j_1} \wedge ... \wedge w_{j_l})$$

Now we have two linear maps that agree on each basis vector so

$$\varphi_{X,Y} \circ (\Lambda^n(\alpha \oplus \beta)) = \left(\bigoplus_{k+l=n} \Lambda^k(\alpha) \otimes \Lambda^l(\beta)\right) \circ \varphi_{V,W}$$

This shows that $\varphi_{(-,-)}$ is a natural transformation between the functors $\Lambda^n(- \oplus -)$ and $\bigoplus_{k+l=n} \Lambda^k(-) \otimes \Lambda^l(-)$.

2

Let R be a Noetherian ring, and let

$$\underline{x} = (x_1, ..., x_n)$$

be a sequence of elements in R. Write $(e_1, ..., e_n)$ be the standard basis of the free R-module R^n. Let $K_\bullet(R, \underline{x})$ be the chain complex

$$R \xleftarrow{\partial_1} \Lambda^1 R^n \xleftarrow{\partial_2} \Lambda^2 R^n \xleftarrow{\partial_3} ... \xleftarrow{\partial_n} \Lambda^n R^n \longleftarrow 0$$

where the exterior powers are taken over R, and $\Lambda^i R^n$ sits in degree i, and the differentials are defined, on the basis discussed in class, as

$$\partial_i(e_{j_1} \wedge e_{j_2} \wedge ... \wedge e_{j_i}) = \sum_{k=1}^{i} (-1)^{k+1} x_{j_k} e_{j_1} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge e_{j_i}$$

a) What is $K_\bullet(R, x)$ for a sequence $\underline{x} = (x)$ of length 1?

$$R \xleftarrow{\partial_1} \Lambda^1 R \longleftarrow 0$$

With $\partial_1 : \Lambda^1 R \to R$ the R-linear map such that $\partial_1(e_1) = x$.

So that with the canonical identification $\Lambda^1 R \cong R$ we have ∂_1 corresponding to multiplication by x.

b) Show that $K_\bullet(R, \underline{x})$ is indeed a chain complex.

$$\delta_{i-1} \circ \delta_i(e_{j_1} \wedge e_{j_2} \wedge ... \wedge e_{j_i}) = \delta_{i-1} \left(\sum_{k=1}^{i} (-1)^{k+1} x_{j_k} e_{j_1} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge e_{j_i} \right)$$

$$= \sum_{k=1}^{i} (-1)^{k+1} x_{j_k} \delta_{i-1}(e_{j_1} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge e_{j_i})$$

$$= \sum_{k=2}^{i} (-1)^{k+1} x_{i_k} \left(\sum_{1 \le l < k} (-1)^{l+1} x_{i_l} e_{j_1} \wedge ... \wedge \widehat{e_{j_l}} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge e_{j_i} \right)$$

$$+ \sum_{k=1}^{i-1} (-1)^{k+1} x_{i_k} \left(\sum_{k < l \le i} (-1)^{l} x_{i_l} e_{j_1} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge \widehat{e_{j_l}} \wedge ... \wedge e_{j_i} \right)$$

$$= \sum_{1\le k<l\le i} (-1)^{k+l+2} x_{i_l} x_{i_k} e_{j_1} \wedge ... \wedge \widehat{e_{j_l}} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge e_{j_i}$$

$$+ \sum_{1\le k<l\le i} (-1)^{k+l+1} x_{i_k} x_{i_l} e_{j_1} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge \widehat{e_{j_l}} \wedge ... \wedge e_{j_i}$$

(Note: that when $k < l$ that e_{j_l} is actually the $(l-1)$-th position as we removed the k-th position where we had e_{j_k}. This is why we only get the $(-1)^{k+l+1}$ instead of the $(-1)^{k+l+2}$.)

$$= \sum_{1\le l<k\le i} (-1)^{k+l+2} x_{i_l} x_{i_k} e_{j_1} \wedge ... \wedge \widehat{e_{j_l}} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge e_{j_i}$$

$$- \sum_{1\le k<l\le i} (-1)^{k+l+2} x_{i_k} x_{i_l} e_{j_1} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge \widehat{e_{j_l}} \wedge ... \wedge e_{j_i}$$

$$= \sum_{1\le l<k\le i} (-1)^{k+l+2} x_{i_l} x_{i_k} e_{j_1} \wedge ... \wedge \widehat{e_{j_l}} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge e_{j_i}$$

$$- \sum_{1\le l<k\le i} (-1)^{k+l+2} x_{i_l} x_{i_k} e_{j_1} \wedge ... \wedge \widehat{e_{j_l}} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge e_{j_i} = 0$$

Where we have simply switched the dummy variables in the second summation.

We have showed that for any $i > 1$ and any basis vector $e_{j_1} \wedge ... \wedge e_{j_i}$ that $\partial_{i-1} \circ \partial_i (e_{j_1} \wedge ... \wedge e_{j_i}) = 0$. This shows that $\partial^2 = 0$ which means that $K_\bullet(R, \underline{x})$ is indeed a chain complex.

(Note: that if we take $\partial_0 = 0$ then trivially $\partial_0 \circ \partial_1 = 0$.)

d) Show that

$$K_\bullet(R, \underline{x}, \underline{y}) = K_\bullet(R, \underline{x}) \otimes_R K_\bullet(R, \underline{y})$$

where $\otimes_R$ is the tensor product of chain complexes.

The definition of the tensor product of chain complexes is given as follows. Let

$$C_0 \xleftarrow{\partial_{c_1}} C_1 \xleftarrow{\partial_{c_2}} C_2 \xleftarrow{\partial_{c_3}} ... \quad \text{and} \quad D_0 \xleftarrow{\partial_{d_1}} D_1 \xleftarrow{\partial_{d_2}} D_2 \xleftarrow{\partial_{d_3}} ...$$

be chain complexes over R. (i.e C_i and D_j are R-modules.

Then

$$C_0 \xleftarrow{\partial_{c_1}} C_1 \xleftarrow{\partial_{c_2}} C_2 \xleftarrow{\partial_{c_3}} ...$$

$$\otimes_R$$

$$D_0 \xleftarrow{\partial_{d_1}} D_1 \xleftarrow{\partial_{d_2}} D_2 \xleftarrow{\partial_{d_3}} ...$$

$$:= \bigoplus_{k+l=0} (C_k \otimes_R D_l) \xleftarrow{\Delta_1} \bigoplus_{k+l=1} (C_k \otimes_R D_l) \xleftarrow{\Delta_2} \bigoplus_{k+l=2} (C_k \otimes_R D_l) \xleftarrow{\Delta_3} ...$$

where

$$\Delta_i : \bigoplus_{k+l=i} (C_k \otimes_R D_l) \to \bigoplus_{k+l=i-1} (C_k \otimes_R D_l) \quad s.t$$

for $x_k \in C_k$ and $y_l \in D_l$

$$\Delta_{k+l}(x_k \otimes y_l) = \partial_{c_k}(x_k) \otimes y_l + (-1)^k x_k \otimes \partial_{d_l}(y_l)$$

Now we will consider the tensor product of $K_\bullet(R, \underline{x}) \otimes_R K_\bullet(R, x_{n+1})$. If we can prove that $K_\bullet(R, \underline{x}) \otimes_R K_\bullet(R, x_{n+1}) = K_\bullet(R, \overline{x}, x_{n+1})$ then by induction we can see that $K_\bullet(R, \underline{x}, \underline{y}) = K_\bullet(R, \underline{x}) \otimes_R K_\bullet(R, \underline{y})$ for any $\underline{y} = (y_1, ..., y_m)$ not just $\underline{y} = x_{n+1}$.

We have the following from the definition of tensor product of chain complexes

$$K_\bullet(R, \underline{x}) \otimes_R K_\bullet(R, x_{n+1}) = R \xleftarrow{D_1} \Lambda^1(R^n) \oplus \Lambda^1(R) \xleftarrow{D_2} \Lambda^2(R^n) \oplus \Lambda^1(R^n) \otimes \Lambda^1(R) \xleftarrow{D_3} ...$$

$$... \xleftarrow{D_i} \big(\Lambda^i(R^n)\big) \oplus \big(\Lambda^{i-1}(R^n) \otimes \Lambda^1(R)\big) \xleftarrow{D_{i+1}} ... \xleftarrow{D_n} (\Lambda^n(R^n)) \oplus \big(\Lambda^{n-1}(R^n) \otimes \Lambda^1(R)\big) \longleftarrow 0$$

where for $a \in \Lambda^i(R^n)$ and $b \otimes e_{n+1} \in \Lambda^{i-1}(R^n) \otimes \Lambda^1(R)$ we have

$$D_i(a + b \otimes e_{n+1}) = \big(\partial_i(a) + (-1)^{i+1} x_{n+1} b\big) + (\partial_{i-1}(b) \otimes e_{n+1})$$

where we have made repeated use of the fact that for A an R-module we have a natural isomorphism $R \otimes_R A \cong A$ and that the chain map for $K_\bullet(R, x_{n+1})$ is simply multiplication by x_{n+1} as was discussed in part a).

We want to show that this chain complex is isomorphic to $K_\bullet(R, \underline{x}, x_{n+1})$.

We must define isomorphisms $\varphi_i : \Lambda^i(R^n) \oplus \Lambda^{i-1}(R^n) \otimes \Lambda^1(R) \to \Lambda^i(R^{n+1})$ such that $\varphi_{i-1} \circ D_i = \partial_i \circ \varphi_i$ that is φ_i such that the following diagram commutes.

$$\begin{array}{ccccccc}
R & \xleftarrow{D_1} & \Lambda^1(R^n) \oplus \Lambda^1(R) & \xleftarrow{D_2} & \Lambda^2(R^n) \oplus \Lambda^1(R^n) \otimes \Lambda^1(R) & \xleftarrow{D_3} & ... \\
\downarrow\varphi_0 & & \downarrow\varphi_1 & & \downarrow\varphi_2 & & \\
R & \xleftarrow{\partial_1} & \Lambda^1(R^{n+1}) & \xleftarrow{\partial_2} & \Lambda^2(R^{n+1}) & \xleftarrow{D_3} & ...
\end{array}$$

$$\begin{array}{ccccc}
... & \xleftarrow{D_{n+1}} & (\Lambda^n(R^n)) \oplus (\Lambda^{n-1}(R^n) \otimes \Lambda^1(R)) & \longleftarrow & 0 \\
 & & \downarrow\varphi_n & & \downarrow \\
... & \xleftarrow{\partial_{n+1}} & \Lambda^{n+1}(R^{n+1}) & \longleftarrow & 0
\end{array}$$

We will use a similar isomorphism we constructed in question 1. Explicitly we consider the basis of $\Lambda^i(R^n)$ and $\Lambda^{i-1}(R^n) \otimes \Lambda^1(R)$ given by $e_{j_1} \wedge ... \wedge e_{j_i} \in \Lambda^i(R^n)$ and $e_{m_1} \wedge ... \wedge e_{m_{i-1}} \otimes e_{n+1} \in \Lambda^{i-1}(R^n) \otimes \Lambda^1(R)$ where $j_1, ..., j_i, m_1, ..., m_{i-1} \in \{1, ..., n\}$ with $j_1 < ... < j_i$ and $m_1 < ... < m_{i-1}$.

Then define φ_i to be the linear map such that $\varphi_i(e_{j_1} \wedge ... \wedge e_{j_i}) = e_{j_1} \wedge ... \wedge e_{j_i} \in \Lambda^i(R^{n+1})$ and $\varphi_i(e_{m_1} \wedge ... \wedge e_{m_{i-1}} \otimes e_{n+1}) = e_{m_1} \wedge ... \wedge e_{m_{i-1}} \wedge e_{n+1}$. We can see that this is an isomorphism as it is induced by a bijective map between the basis vectors of a free R-module.

Now we wish to show that $\varphi_{i-1} \circ D_i = \partial_i \circ \varphi_i$ on each of the basis vectors.

$$\varphi_{i-1} \circ D_i(e_{j_1} \wedge ... \wedge e_{j_i}) = \varphi_{i-1}(\delta_i(e_{j_1} \wedge ... \wedge e_{j_i}))$$

$$= \varphi_{i-1}\left(\sum_{k=1}^{i}(-1)^{k+1}x_{j_k}e_{j_1} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge e_{j_i}\right)$$

$$= \sum_{k=1}^{i}(-1)^{k+1}x_{j_k}\varphi_{i-1}(e_{j_1} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge e_{j_i})$$

$$= \sum_{k=1}^{i}(-1)^{k+1}x_{j_k}e_{j_1} \wedge ... \wedge \widehat{e_{j_k}} \wedge ... \wedge e_{j_i}$$

$$= \partial_i(e_{j_1} \wedge ... \wedge e_{j_i}) = \partial_i \circ \varphi_i(e_{j_1} \wedge ... \wedge e_{j_i})$$

We also have

$$\varphi_{i-1} \circ D_i(e_{m_1} \wedge ... \wedge e_{m_{i-1}} \otimes e_{n+1})$$

$$= \varphi_{i-1}(\partial_{i-1}(e_{m_1} \wedge ... \wedge e_{m_{i-1}}) \otimes e_{n+1} + (-1)^{i+1}e_{m_1} \wedge ... \wedge e_{m_{i-1}} \otimes \partial(e_{n+1}))$$

$$= \varphi_{i-1}\left(\sum_{k=1}^{i-1}(-1)^{k+1}x_{m_k}e_{m_1} \wedge ... \wedge \widehat{e_{m_k}} \wedge ... \wedge e_{m_{i-1}} \otimes e_{n+1} + (-1)^{i+1}x_{n+1}e_{m_1} \wedge ... \wedge e_{m_{i-1}}\right)$$

$$= \sum_{k=1}^{i-1}(-1)^{k+1}x_{m_k}\varphi_{i-1}\left(e_{m_1} \wedge ... \wedge \widehat{e_{m_k}} \wedge ... \wedge e_{m_{i-1}} \otimes e_{n+1}\right) + (-1)^{i+1}x_{n+1}\varphi\left(e_{m_1} \wedge ... \wedge e_{m_{i-1}}\right)$$

$$= \sum_{k=1}^{i-1}(-1)^{k+1}x_{m_k}e_{m_1} \wedge ... \wedge \widehat{e_{m_k}} \wedge ... \wedge e_{m_{i-1}} \wedge e_{n+1} + (-1)^{i+1}x_{n+1}e_{m_1} \wedge ... \wedge e_{m_{i-1}}$$

$$= \sum_{k=1}^{i}(-1)^{k+1}x_{m_k}e_{m_1} \wedge ... \wedge \widehat{e_{m_k}} \wedge ... \wedge e_{m_i}$$

where we denote $m_i = n + 1$.

$$= \partial_i(e_{m_1} \wedge ... \wedge e_{m_i}) = \partial_i \circ \varphi_i(e_{m_1} \wedge ... \wedge e_{m_{i-1}} \otimes e_{m_i}) = \partial_i \circ \varphi_i(e_{m_1} \wedge ... \wedge e_{m_{i-1}} \otimes e_{n+1})$$

So for all basis vectors $e_{j_1} \wedge ... \wedge e_{j_i} \in \Lambda^i(R^n)$ and $e_{m_1} \wedge ... \wedge e_{m_{i-1}} \otimes e_{n+1} \in \Lambda^{i-1}(R^n) \otimes \Lambda^1(R)$ we have

$$\varphi_{i-1} \circ D_i(e_{j_1} \wedge ... \wedge e_{j_i}) = \partial_i \circ \varphi_i(e_{j_1} \wedge ... \wedge e_{j_i})$$

$$\varphi_{i-1} \circ D_i(e_{m_1} \wedge ... \wedge e_{m_{i-1}} \otimes e_{n+1}) = \partial_i \circ \varphi_i(e_{m_1} \wedge ... \wedge e_{m_{i-1}} \otimes e_{n+1})$$

As $\varphi_{i-1} \circ D_i$ and $\partial_i \circ \varphi_i$ are linear maps we see that $\varphi_{i-1} \circ D_i = \partial_i \circ \varphi_i$.

This shows that φ_i is an isomorphism between $K_\bullet(R, \underline{x}, x_{n+1}) = K_\bullet(R, \underline{x}) \otimes_R K_\bullet(R, x_{n+1})$. By induction we have $K_\bullet(R, \underline{x}, \underline{y}) \cong K_\bullet(R, \underline{x}) \otimes_R K_\bullet(R, \underline{y})$.

We also get by induction that

$$K_\bullet(R, \underline{x}) \cong K_\bullet(R, x_1) \otimes_R ... \otimes_R K_\bullet(R, x_n)$$

Using a) we see that the i-th degree of the RHS is given by

$$\bigoplus_{i_1,...,i_n \in \{0,1\}}^{i_1+...+i_n=i} \Lambda^{i_1}(R) \otimes_R ... \otimes_R \Lambda^{i_n}(R) \cong \bigoplus_{i_1,...,i_n \in \{0,1\}}^{i_1+...+i_n=i} R \cong R^{\binom{n}{k}}$$

where we take $\Lambda^0(R) = R$. If we let $e_{j_1...j_i}$ be the basis vector for the $j_1, ..., j_i$ term in the summation. We get the chain map

$$D_i(e_{j_1...j_i}) = \sum_{k=1}^{i} (-1)^{k+1} x_{j_k} e_{j_1...\widehat{e_{j_k}}...j_i}$$

Note that this means our complex is made of out free R-modules.

c) Assuming that $\underline{x}$ is an R-regular sequence, show that $K_\bullet(R, \underline{x})$ is a free resolution of $R/(\underline{x})$.

A sequence $(x_1, ...x_n)$ is R-regular if and only if for $i \in 1, ..., n$ that $x_i + (x_1, ..., x_{i-1}) \in R/(x_1, ..., x_i)$ is not a 0 divisor where for $i = 1$ we mean $x_1 \in R$ is not a zero divisor.

Firstly note that R and $\Lambda^i(R^n)$ are both free R-modules (this was made explicit at the end of d)) and so if we show that

$$0 \longleftarrow R/(x_1, ..., x_n) \xleftarrow{q_n} K_\bullet(R, x_1, ..., x_n)$$

is exact. Then by definition $K_\bullet(R, x_1, ..., x_n)$ is a free resolution of $R/(x_1, ..., x_n)$. We will show this by induction. Our base case is given as follows.

$$0 \longleftarrow R/(x_1) \xleftarrow{q_1} R \xleftarrow{\partial_1} R \longleftarrow 0$$

Our inductive step will be the following. If

$$0 \longleftarrow R/(\underline{x}) \xleftarrow{q_n} R \xleftarrow{\partial_1} \Lambda^1 R^n \xleftarrow{\partial_2} \Lambda^2 R^n \xleftarrow{\partial_3} ... \xleftarrow{\partial_n} \Lambda^n R^n \longleftarrow 0$$

is exact then

$$0 \longleftarrow R/(\underline{x}) \xleftarrow{q_{n+1}} R \xleftarrow{\partial_1} \Lambda^1 R^{n+1} \xleftarrow{\partial_2} \Lambda^2 R^{n+1} \xleftarrow{\partial_3} ... \xleftarrow{\partial_{n+1}} \Lambda^{n+1} R^{n+1} \longleftarrow 0$$

is exact. The result will then follow by induction.

Base Case: We will prove the base case. Consider

$$0 \longleftarrow R/(x_1) \xleftarrow{q_1} R \xleftarrow{\partial_1} R \longleftarrow 0$$

Then notice that $q_1 \circ \partial_1(r) = q_1(x_1 r) = x_1 r + (x_1) = 0 + (x_1)$.

So $0 \longleftarrow R/(x_1) \xleftarrow{q_1} R \xleftarrow{\partial_1} R \longleftarrow 0$ is indeed as chain complex.

Now note that ∂_1 is injective as for $r \in R$ we have $0 = \partial_1(r) = x_1 r$. We know that (x_1) is an R-regular sequence so in particular x_1 is not a zero divisor. This means that $r = 0$ which in turn means that ∂_1 is injective.

Now we can also see that q_1 is surjective as it is a quotient map.

This shows that $0 \longleftarrow R/(x_1) \xleftarrow{q_1} R \xleftarrow{\partial_1} R \longleftarrow 0$ is a short exact sequence and therefore we have that

$$0 \longleftarrow R/(x_1) \xleftarrow{q_1} K_\bullet(R, x_1)$$

is exact and therefore we have that $K_\bullet(R, x_1)$ is a free resolution of $R/(x_1)$.

Inductive Step: Let

$$0 \longleftarrow R/(\underline{x}) \xleftarrow{q_n} R \xleftarrow{\partial_1} \Lambda^1 R^n \xleftarrow{\partial_2} \Lambda^2 R^n \xleftarrow{\partial_3} \ldots \xleftarrow{\partial_n} \Lambda^n R^n \longleftarrow 0$$

be an exact chain complex. Consider

$$0 \longleftarrow R/(\underline{x}, x_{n+1}) \xleftarrow{q_{n+1}} R \xleftarrow{\partial_1} \Lambda^1 R^{n+1} \xleftarrow{\partial_2} \Lambda^2 R^{n+1} \xleftarrow{\partial_3} \ldots \xleftarrow{\partial_{n+1}} \Lambda^{n+1} R^{n+1} \longleftarrow 0$$

From question d) we have

$$0 \longleftarrow R/(\underline{x}, x_{n+1}) \xleftarrow{q_{n+1}} R \xleftarrow{\partial_1} \Lambda^1 R^{n+1} \xleftarrow{\partial_2} \Lambda^2 R^{n+1} \xleftarrow{\partial_3} \ldots \xleftarrow{\partial_{n+1}} \Lambda^{n+1} R^{n+1} \longleftarrow 0$$

$$\cong 0 \longleftarrow R/(\underline{x}, x_{n+1}) \xleftarrow{q_{n+1}} R \xleftarrow{D_1} \Lambda^1(R^n) \oplus \Lambda^1(R) \xleftarrow{D_2} \Lambda^2(R^n) \oplus \Lambda^1(R^n) \otimes \Lambda^1(R) \xleftarrow{D_3} \ldots$$

$$\ldots \xleftarrow{D_i} \big(\Lambda^i(R^n)\big) \oplus \big(\Lambda^{i-1}(R^n) \otimes \Lambda^1(R)\big) \xleftarrow{D_{i+1}} \ldots \xleftarrow{D_n} \big(\Lambda^n(R^n)\big) \oplus \big(\Lambda^{n-1}(R^n) \otimes \Lambda^1(R)\big) \longleftarrow 0$$

There are four cases of interest. Need to prove exactness at $R/(\underline{x}, x_{n+1})$, R, $\Lambda^1(R^n) \oplus \Lambda^1(R)$ and $(\Lambda^i(R^n)) \oplus (\Lambda^{i-1}(R^n) \otimes \Lambda^1(R))$. Cases 1 and 2 and part d) prove that we have a chain complex.

Case 1: $R/(\underline{x}, x_{n+1})$

$ker(0) = R/(\underline{x}, x_{n+1})$ and as q_m is a quotient map and therefore surjective $Im(q_{n+1}) = R/(\underline{x}, x_{n+1})$.

So $ker(0) = Im(q_{n+1})$.

Case 2: R

$ker(q_{n+1}) = (x_1, ..., x_{n+1})$.

So $Im(D_1) = \left\{D_1(\sum_{i=1}^{n+1} r_i e_i) | r_i \in R\right\} = \left\{\sum_{i=1}^{n+1} r_i D_1(e_i) | r_i \in R\right\} = \left\{\sum_{i=1}^{n+1} r_i x_i | r_i \in R\right\} = (x_1, ..., x_{n+1})$.

So $ker(q_{n+1}) = Im(D_1)$.

Case 3: $\Lambda^1(R^n) \oplus \Lambda^1(R)$

We know that $D_1 \circ D_2 = 0$ so we see that $Im(D_2) \subseteq ker(D_1)$.

We want to show that $ker(D_1) \subseteq Im(D_2)$.

Let $D_1(\sum_{i=1}^{n+1} r_i e_i) = \sum_{i=1}^{n+1} r_i x_i = 0$ or in other words $\sum_{i=1}^{n+1} r_i e_i \in ker(D_1)$. We want to show that $\sum_{i=1}^{n+1} r_i e_i \in Im(D_2)$.

Now consider the expression $0 = D_1(\sum_{i=1}^{n+1} r_i e_i)$ in the following quotient

$$0 + (x_1, ..., x_n) = \sum_{i=1}^{n+1} r_i x_i + (x_1, ..., x_n) = r_{n+1} x_{n+1} + (x_1, ..., x_n)$$

As $x_1, ..., x_{n+1}$ is R-regular x_{n+1} is not a zero divisor in $R/(x_1, ..., x_n)$ so $r_{n+1} + (x_1, ..., x_n) = 0 + (x_1, ..., x_n)$. In other words

$$r_{n+1} \in (x_1, ..., x_n)$$

This means $r_{n+1} = \sum_{i=1}^{n} s_i x_i$ for some $s_i \in R$. So we get

$$\sum_{i=1}^{n+1} r_i e_i = \left(\sum_{i=1}^{n} r_i e_i\right) + \sum_{i=1}^{n} s_i x_i e_{n+1}$$

and also

$$0 = \sum_{i=1}^{n+1} r_i x_i = \sum_{i=1}^{n} (r_i + s_i x_{n+1}) x_i$$

So

$$\partial_1(\sum_{i=1}^{n} (r_i + s_i x_{n+1}) e_i) = 0$$

This means by our inductive assumption, $ker(\partial_1) = Im(\partial_2)$, that there exists $p \in \Lambda^2(R^n)$ such that

$$\partial_2(p) = \sum_{i=1}^{n} (r_i + s_i x_{n+1}) e_i$$

Consider $p + \sum_{i=1}^{n} s_i e_i \otimes e_{n+1} \in \Lambda^1(R^n) \oplus \Lambda^1(R)$. Now note that

$$D_1\left(p + \sum_{i=1}^{n} s_i e_i \otimes e_{n+1}\right) = \partial_2(p) + \partial_1\left(\sum_{i=1}^{n} s_i e_i\right) \otimes e_{n+1}(-1)^{2+1} \sum_{i=1}^{n} s_i x_{n+1} e_i$$

$$= \sum_{i=1}^{n}(r_i + s_i x_{n+1})e_i + \sum_{i=1}^{n} s_i x_i e_{n+1} - \sum_{i=1}^{n} s_i x_{n+1} e_i$$

$$= \sum_{i=1}^{n} r_i e_i + r_{n+1} e_{n+1} = \sum_{i=1}^{n+1} r_i e_i$$

where we used the fact that the $s_i \in R$ were defined such that $\sum_{i=1}^{n} s_i x_i = r_{n+1}$. This shows that $ker(D_1) \subseteq Im(D_2)$ and therefore that $ker(D_1) = Im(D_2)$.

Case 4: $(\Lambda^i(R^n)) \oplus (\Lambda^{i-1}(R^n) \otimes \Lambda^1(R))$ for $2 \leq i \leq n$ (we use ∂_{i-1} which means that $i > 1$ so this does not apply to case 3).

We know from b) and d) that $D_i \circ D_{i+1} = 0$. This means that $Im(D_{i+1}) \subseteq ker(D_i)$. So we must know show that $Im(D_{i+1}) \supseteq ker(D_i)$.

Let $(a + b \otimes e_{n+1}) \in (\Lambda^i(R^n)) \oplus (\Lambda^{i-1}(R^n) \otimes \Lambda^1(R))$ such that $D_i(a + b \otimes e_{n+1}) = 0$. That is let $a + b \otimes e_{n+1} \in ker(D_i)$.

$$D_i(a + b \otimes e_{n+1}) = \big(\partial_i(a) + (-1)^{i+1} x_{n+1} b\big) + (\partial_{i-1}(b) \otimes e_{n+1}) = 0$$

$$\Rightarrow \partial_i(a) + (-1)^{i+1} x_{n+1} b = 0 \quad \text{and} \quad \partial_{i-1}(b) = 0$$

As we have assumed for our induction that $0 \longleftarrow R/(x_1, ..., x_n) \overset{q_n}{\longleftarrow} K_\bullet(R, x_1, ..., x_n)$ is exact we have $ker(\partial_{i-1}) = Im(\partial_i)$ and $ker(\partial_i) = Im(\partial_{i+1})$. Therefore there exists $q \in \Lambda^i(R^n)$ such that $\partial_i(q) = b$. Therefore

$$0 = \partial_i(a) + (-1)^{i+1} x_{n+1} b = \partial_i(a) + (-1)^{i+1} x_{n+1} \partial_i(q) = \partial_i(a + (-1)^{i+1} x_{n+1} q)$$

So $a + (-1)^{i+1} x_{n+1} q \in ker(\partial_i) = Im(\partial_{i+1})$.

So there exists $p \in \Lambda^{i+1}(R^n)$ such that $\partial_{i+1}(p) = a + (-1)^{i+1} x_{n+1} q$.
Consider $p + q \otimes e_{n+1} \in (\Lambda^{i+1}(R^n)) \oplus (\Lambda^i(R^n) \otimes \Lambda^1(R))$.

$$D_{i+1}(p + q \otimes e_{n+1}) = \partial_{i+1}(p) + (-1)^{i+2} x_{n+1} q + \partial_i(q) \otimes e_{n+1}$$

$$= a + (-1)^{i+1} x_{n+1} q + (-1)^{i+2} x_{n+1} q + b \otimes e_{n+1}$$

$$= a + (-1)^{i+1} x_{n+1} q - (-1)^{i+1} x_{n+1} q + b \otimes e_{n+1} = a + b \otimes e_{n+1}$$

This shows that $a + b \otimes e_{n+1} \in Im(D_{i+1})$. This means that $ker(D_i) \subseteq Im(D_{i+1})$.

Therefore $ker(D_i) = Im(D_{i+1})$ for all $2 \leq i \leq n + 1$

This covers all cases in the induction and completes the proof.

3

Let X be a compact Hausdorff space and let $C(X)$ denote the ring of all real-valued continuous function on X.

Let $x \in X$ and $\mathfrak{m}_x = \{f \in C(X) | f(x) = 0\}$.

We claim that $\mathfrak{m}_x$ is maximal.
We have the following short exact sequence where the first arrow is inclusion and the second arrow is evaluation at x.

$$0 \to \mathfrak{m}_x \hookrightarrow C(X) \twoheadrightarrow \mathbb{R} \to 0$$

So $C(X)/\mathfrak{m}_x \cong \mathbb{R}$. We know if a ring quotient an ideal is a field that the ideal must be maximal.
Therefore $\mathfrak{m}_x$ is maximal.

Denote $Max(C(X))$ by $\widetilde{X}$.
Define $\mu : X \to \widetilde{X}$ such that $\mu(x) = \mathfrak{m}_x$.
We claim that this is a homeomorphism between X and $\widetilde{X}$.

Injectivity: First to show injectivity we need to use Urysohn's Lemma noting that X is compact Hausdorff.
Urysohn's lemma says for $x \neq y \in X$ there exists $f \in C(X, [0, 1])$ such that $f(x) = 1$ and $f(y) = 0$. So $f \in \mathfrak{m}_y$ but $f \notin \mathfrak{m}_x$. Therefore if $x \neq y$ then $\mu(x) = \mathfrak{m}_x \neq \mathfrak{m}_y = \mu(y)$.
Therefore μ is injective.

Surjectivity: We will now show that μ is surjective.
Let $\mathfrak{m} \in \widetilde{X}$.
Define the following subset

$$V(\mathfrak{m}) = \{x \in X | \forall f \in \mathfrak{m} \quad f(x) = 0\}$$

There are two cases $V = \emptyset$ and $V \neq \emptyset$.
Suppose that $V = \emptyset$.
Then for each $x \in X$ there exists $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$.
As $f_x(x) \neq 0$ and f_x is continuous there exists an open neighbourhood $x \in U_x$ such that $0 \notin f_x(U_x)$.
Consider the cover $\bigcup_{x \in X} U_x = X$.
As X is compact there exists a finite subcover.
That is there exists points $x_1, ..., x_n \in X$ such that $\bigcup_{i=1}^n U_{x_i} = X$.
Consider the function $f = \sum_{i=1}^n (f_{x_i})^2 : X \to \mathbb{R}_{\geq 0}$.
For $x \in X$ we have $x \in U_{x_j}$ for some $j = 1, ..., n$.
So $0 < f_{x_j}(x)^2 \leq \sum_{i=1}^n (f_{x_i})(x)^2 = f(x)$.

This means that $\forall x \in X \; f(x) \neq 0$ which means that $\frac{1}{f}$ is continuous.
Therefore $1 = \frac{1}{f} f \in \mathfrak{m}$ and therefore $\mathfrak{m} = C(X)$ which contradicts that fact that $\mathfrak{m}$ is maximal.
So the only possible case is that $V(\mathfrak{m}) \neq \emptyset$.
Suppose $x \in V(\mathfrak{m})$.
Then $\forall f \in \mathfrak{m}$ we have $f(x) = 0$.
So $\mathfrak{m} \subseteq \mathfrak{m}_x$.
As $\mathfrak{m}$ and $\mathfrak{m}_x$ are maximal we have $\mathfrak{m} = \mathfrak{m}_x$.
This show that if $\mathfrak{m} \in \widetilde{X}$ that there exists $x \in X$ such that $\mathfrak{m} = \mathfrak{m}_x = \mu(x)$.
In other words μ is surjective.

Continuous and Open: Again using Urysohn's lemma we get the following basis for the topology of X given all $f \in C(X)$.

$$U_f = \{x \in X | f(x) \neq 0\}$$

The standard topology on $\widetilde{X}$ is given by the following basis given all $f \in C(X)$.

$$V_f = \{\mathfrak{m} \in \widetilde{X} | f \notin \mathfrak{m}\}$$

Consider the following.

$$\mu(U_f) = \mu(\{x \in X | f(x) \neq 0\}) = \{\mu(x) \in \widetilde{X} | x \in X \text{ and } f(x) \neq 0\}$$
$$= \{\mathfrak{m}_x \in \widetilde{X} | x \in X \text{ and } f(x) \neq 0\} = \{\mathfrak{m}_x \in \widetilde{X} | x \in X \text{ and } f \notin \mathfrak{m}_x\}$$
$$= \{\mathfrak{m} \in \widetilde{X} | f \notin \mathfrak{m}\} = V_f$$

where we used surjectivity of μ in the second last equality.
So $\mu(U_f) = V_f$.
This means that μ maps the basis $\{U_f\}$ for the topology of X to the basis $\{V_f\}$ for the topology of $\widetilde{X}$.
So μ is open.
As μ is a bijection we see that $\mu^{-1}(V_f) = U_f$ and therefore μ is continuous by the same reasoning.

So μ is a homeomorphism.

4

Let $\mathbf{k}$ be an algebraically closed field.
Consider the polynomial ring in n variables $\mathbf{k}[t_1, ..., t_n]$.
Let $A \subseteq \mathbf{k}[t_1, ..., t_n]$.
Define

$$V(A) = \{x \in \mathbf{k}^n | \forall f \in A \quad f(x) = 0\}$$

were we identify $f \in \mathbf{k}[t_1, ..., t_n]$ as polynomial functions from $\mathbf{k}^n$ to $\mathbf{k}$.
$V(A)$ is what we call an affine variety.

Let $I(V(A)) = \{g \in \mathbf{k}[t_1, ..., t_n] | \forall x \in V(A) \quad g(a) = 0\}$.
We call $I(V(A))$ the ideal of the variety $V(A)$.
(Note: this is a subspace and for $f \in \mathbf{k}[t_1, ..., t_n]$ and $g \in I(V(A))$ we see that for $x \in A$ we have $(f \cdot g)(x) = f(x)g(x) = f(x) \cdot 0 = 0$. So $I(V(A))$ is in fact an ideal.)

Define the following ring

$$P(V(A)) = \mathbf{k}[t_1, ..., t_n]/I(V(A))$$

We can view $P(V(A))$ as polynomial functions on $V(A)$ as for $f, g \in \mathbf{k}[t_1, ..., t_n]$ then $f + I(V(A)) = g + I(V(A))$ if and only if $f - g + I(V(A)) = 0 + I(V(A))$ if and only if $f - g \in I(V(A))$ if and only if for $x \in V(A)$ we have $f(x) - g(x) = 0$ and therefore $f(x) = g(x)$.

Let $\xi_i = t_i + I(V(A)) \in P(V(A))$.
We call the ξ_i coordinate functions on $V(A)$.
As a $\mathbf{k}$ algebra $\mathbf{k}[t_1, ..., t_n]$ is generated by the set $\{t_1, ..., t_n\}$.
So as a $\mathbf{k}$ algebra $\{\xi_1, ..., \xi_n\}$ generate $P(V(A))$ as $P(V(A))$ is a quotient of $\mathbf{k}[t_1, ..., t_n]$ and the ξ_i are the images of the generators of $\mathbf{k}[t_1, ..., t_n]$ under the quotient map.

Let $x \in V(A)$ and define $\mathfrak{m}_x = \{f \in P(V(A)) | f(x) = 0\}$.

(Note: that evaluating $f \in P(V(A))$ on $V(A)$ makes sense as if we have some $g \in I(V(A))$ and consider $f + g \in f + I(V(A))$ then $(f + g)(x) = f(x) + g(x)$ but by definition for $x \in V(A)$ we have $g(x) = 0$ and so $(f + g)(x) = f(x)$. So evaluating f doesn't depend on the choice of representative of the equivalence class.)

We claim that $\mathfrak{m}_x$ is maximal.
We have that following short exact sequence where the first arrow is inclusion and the second is evaluation at $x \in V(A)$ which we discussed was well defined.

$$0 \to \mathfrak{m}_x \hookrightarrow P(V(A)) \twoheadrightarrow \mathbf{k} \to 0$$

So $P(V(A))/\mathfrak{m}_x \cong \mathbf{k}$. We know if a ring quotient an ideal is a field that the ideal must be maximal.
Therefore $\mathfrak{m}_x$ is maximal.

Let $\widetilde{V(A)} = Max(P(V(A)))$.
Define $\mu : V(A) \to \widetilde{V(A)}$ such that $\mu(x) = \mathfrak{m}_x$.
We claim that μ is a bijection.

Injectivity: Let $(x_1, ..., x_n) = x \neq y = (y_1, ..., y_n) \in V(A)$. Then there exists $i \in \{1, ..., n\}$ such that $x_i \neq y_i$.
Consider the $\xi_i - x_i \in I(V(A))$.
Then $(\xi_i - x_i)(x) = \xi_i(x) - x_i = x_i - x_i = 0$.
But $(\xi_i - x_i)(y) = \xi_i(y) - x_i = y_i - x_i \neq 0$.
So $\xi_i - x_i \in \mathfrak{m}_x$ but $\xi_i - y \notin \mathfrak{m}_y$.
So $\mathfrak{m}_x \neq \mathfrak{m}_y$.

Surjectivity: Let $\pi : \mathbf{k}[t_1, ..., t_n] \to P(V(A))$ be the quotient map.
Let $\mathfrak{m} \lhd P(V(A))$ be maximal.
Then consider $\pi^{-1}(\mathfrak{m})$.
As $\mathfrak{m}$ is maximal it is prime.
This means that $\pi^{-1}(\mathfrak{m})$ is prime but not necessarily maximal.
Note that $\pi^{-1}(0) = I(V(A)) \subseteq \pi^{-1}(\mathfrak{m})$.

Consider $V(\pi^{-1}(\mathfrak{m}))$.
From Hilbert's Nullsetellensatz we know that

$$I(V(\pi^{-1}(\mathfrak{m}))) = \sqrt{\pi^{-1}(\mathfrak{m})} = \pi^{-1}(\mathfrak{m})$$

where the last inequality if gotten noting that $\pi^{-1}(\mathfrak{m})$ is prime.

There are two cases for $V(\pi^{-1}(\mathfrak{m}))$ either $V(\pi^{-1}(\mathfrak{m})) = \emptyset$ or $V(\pi^{-1}(\mathfrak{m})) \neq \emptyset$.
Note that $I(\emptyset) = \mathbf{k}[t_1, ..., t_n]$ as $1 \in I(\emptyset)$.
So if $V(\pi^{-1}(\mathfrak{m})) = \emptyset$ then $\pi^{-1}(\mathfrak{m}) = I(\emptyset) = \mathbf{k}[t_1, ..., t_n]$.
This means that $\mathfrak{m} = \pi(\pi^{-1}(\mathfrak{m})) = \pi(\mathbf{k}[t_1, ..., t_n]) = P(V(A))$.
This contradicts the fact that $\mathfrak{m}$ is maximal.
So there exists $x \in V(\pi^{-1}(\mathfrak{m}))$.

Let $x \in V(\pi^{-1}(\mathfrak{m}))$.
Now as $I(V(A)) \subseteq \pi^{-1}(\mathfrak{m})$ we can see that $\forall f \in I(V(A))$ we have $f(x) = 0$.
In particular noting that $A \subseteq I(V(A))$ we have $\forall f \in A$ we have $f(x) = 0$.
This means that $x \in V(A)$.

So $\forall f \in \mathfrak{m}$ we have $(f + I(V(A)))(x) = f(x) + I(V(A))(x) = 0 + 0 = 0$.
This is well defined as $x \in V(A)$.

So $\mathfrak{m} \subseteq \mathfrak{m}_x$.
So as $\mathfrak{m}$ and $\mathfrak{m}_x$ are maximal we have $\mathfrak{m} = \mathfrak{m}_x = \mu(x)$.

So for any $\mathfrak{m} \in \widetilde{V(A)}$ there exists an $x \in V(A)$ such that $\mathfrak{m} = \mu(x)$.
That is μ is surjective.

So μ is a bijection.