Assigment 1

(1) Show that the symmetric and the exterior powers are exponential in the following sense. There exist natural isomorphisms

(a)
$$S^n(V \oplus W) \cong \bigoplus_{k+l=n} S^k V \otimes S^l W$$
and

(b)
$$\Lambda^n(V \oplus W) \cong \bigoplus_{k+l=n} \Lambda^k V \otimes \Lambda^l W.$$

Here V and W are vector spaces over a fixed field, $\mathbf{k}$ and the tensor product and direct sum are taken over $\mathbf{k}$. Only if you like, you may imagine V and W to be something more interesting, say, vector bundles over a manifold M or representations of a group G.

(2) Let R be a Noetherian ring, and let

$$\underline{x} = (x_1, \dots, x_n)$$

be a sequence of elements in R. Write $(e_1, \dots, e_n)$ be the standard basis of the free R-module R^n. Let $K_\bullet(R, \underline{x})$ be the chain complex

$$R \xleftarrow[\partial_1]{} \Lambda^1 R^n \xleftarrow[\partial_2]{} \Lambda^2 R^n \xleftarrow[\partial_3]{} \dots \xleftarrow[\partial_n]{} \Lambda^n R^n \longleftarrow 0,$$

where the exterior powers are taken over R, and $\Lambda^i R^n$ sits in degree i, and the differentials are defined, on the basis discussed in class, as

$$\partial_i(e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_i}) = \sum_{k=1}^{i} (-1)^{k+1} x_{j_k} e_{j_1} \wedge \dots \wedge \widehat{e}_{j_k} \wedge \dots \wedge e_{j_i}.$$

(a) What is $K_\bullet(R, x)$ for a sequence $\underline{x} = (x)$ of length 1?

(b) Show that $K_\bullet(R, \underline{x})$ is indeed a chain complex.

(c) Assuming that $\underline{x}$ is an R-regular sequence (what is that?), show that $K_\bullet(R, \underline{x})$ is a free resolution of $R/(\underline{x})$.

(d) Show that
$$K_\bullet(R, \underline{x}, \underline{y}) = K_\bullet(R, \underline{x}) \otimes_R K_\bullet(R, \underline{y}).$$
(tensor product of chain complexes, careful with the sign convention), and use this to express $K_\bullet(R, \underline{x})$ in terms of your answer to (a).

(3) Atiyah and Macdonald Exercise 26

(4) Atiyah and Macdonald Exercise 27