Assigment 2

(1) Let A be a ring, and let $\mathfrak{p}$ be a prime ideal in A.
 (a) Show that the extended ideal $(\mathfrak{p}^n)_{\mathfrak{p}}$ is primary in $A_{\mathfrak{p}}$. What is its radical?
 (b) Assume that we are given a primary decomposition

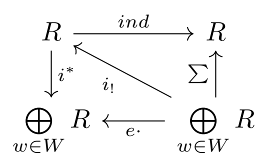
$$\mathfrak{p}^n = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$$

 in reduced form. Show that exactly one of the $\mathfrak{q}_i$ is contained in $\mathfrak{p}$. After possibly reordering, we may take this to be $\mathfrak{q}_1$.
 (c) Show that we have an equality of ideals $(\mathfrak{p}^n)_{\mathfrak{p}} = (\mathfrak{q}_1)_{\mathfrak{p}}$ in $A_{\mathfrak{p}}$.
 (d) Show that $\mathfrak{q}_1$ is the largest $\mathfrak{p}$-primary ideal in A with the property you proved in (c).

(2) Let Λ be a free $\mathbb{Z}$-module of rank d, and consider the ring

$$R = \mathbb{Z}\{X^\lambda\}_{\lambda\in\Lambda} \qquad X^\lambda X^\mu = X^{\lambda+\mu},$$

and write 1 for X^0. Write $e(\lambda) = 1 - X^\lambda$ and assume we are given a finite set $\alpha_1, \ldots, \alpha_n$ of non-zero elements of Λ and a finite group W of automorphisms of Λ.
 (a) Let S be the smallest multiplicative set containing the elements $e(\alpha_1), \ldots, e(\alpha_n)$. Show that the map from R to its localization at S is injective.
 (b) **challenge question, won't be marked** You are given the information that there is a commuting diagram



 and the information that $i^*(X^\lambda) = (X^{w(\lambda)})_{w\in W}$, and $e\cdot = \bigoplus_{w\in W} e_w\cdot$, which on the wth summand is multiplication by $e_w = \prod_{i=1}^n e(w\alpha_i)$ and $\sum$ is given by the universal property of the direct sum applied to the identity on each summand, while the remaining two maps are a mystery. Deduce that $ind(X^\lambda)$ is given by the Weyl character formula.

(3) Let A be a commutative ring with 1, and let $\mathfrak{a} \subset A$ be an ideal, let $R = A/\mathfrak{a}$, and let $f \in R$.
 (a) Find a homeomorphism from

$$U_f = spec(R_f)$$

 to an open subset of $spec(R)$.
 (b) Make U_f explicit in the example $A = \mathbb{C}[x]$ and $\mathfrak{a} = 0$ and $f = x$.

(4) Atiyah-Macdonald, Chapter 3, Exercises 12 and 13.