Open key encryption after Rivest-Shamir-Adleman

A contribution to the MUMS student seminar series

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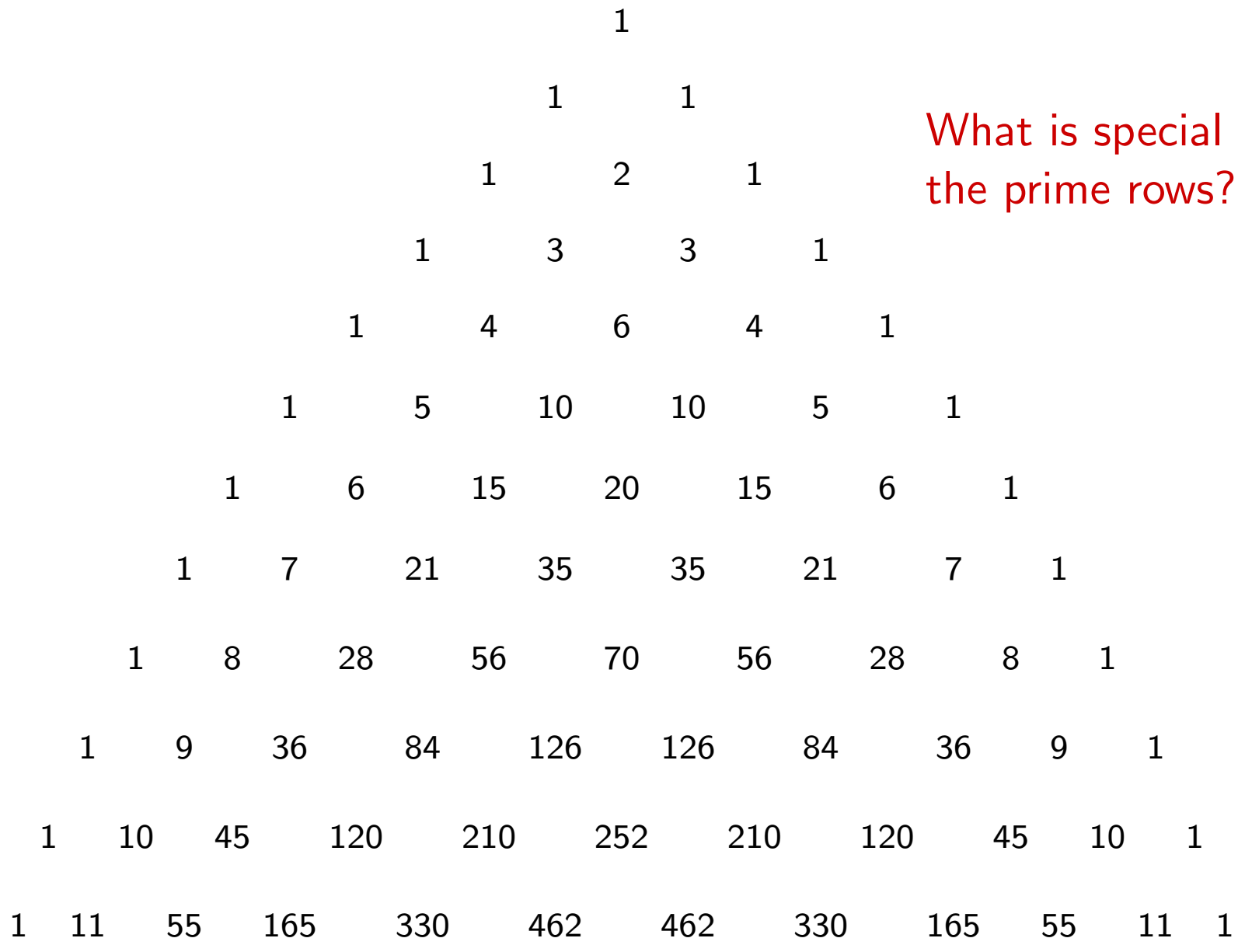
Prime Rows in the Pascal Triangle



What is special about the prime rows?

Divisibility of prime rows

The pth row in the Pascal triangle lists the numbers

$$\binom{p}{k} = \frac{p \times (p-1) \times \cdots \times (p-k+1)}{k \times (k-1) \times \cdots \times 1}$$

The simplified fraction is a natural number, namely the number of possibilities to choose k objects out of a set of p.

The number p divides the numerator but not the denominator. If p is a prime number, it follows that it divides $\binom{p}{k}$, too.

Here we assumed $1 \leq k \leq p-1$ and used, without proof:

Euler's Lemma: If a prime number p divides a product, then it divides at least one of the factors.

Fermat's little theorem

Fix a prime p and let a be a natural number. Then

$$a^p \equiv a \mod p,$$

i.e., the remainders of a^p and a when dividing by p are equal.

Proof by induction, starting at $a = 0$.

Multiplication modulo p

Let a be an integer, and let p be a prime number.

Multiplication with a modulo p is a one-to-one correspondence:

$$\begin{array}{rcl} 0 & \longleftrightarrow & 0 \\ 1 & \longleftrightarrow & a \\ 2 & \longleftrightarrow & 2 \times a \\ & \vdots & \\ p-1 & \longleftrightarrow & (p-1) \times a. \end{array}$$

Proof: Euler's Lemma.

Multiplicative inverses modulo p

If p is understood, we write $a^{-}1$ for any natural number with

$$a \times a^{-1} \equiv 1 \equiv a^{-1} \times a \qquad \text{mod } p.$$

This characterises a^{-1} uniquely modulo p. Modulo p, multiplication with a^{-1} is inverse to multiplication with a.

Example: What is the inverse of 3 modulo 7?

This property makes the set $\{[0], [1], \ldots [p-1]\}$ of remainder classes, endowed with addtion and multiplication modulo p, a field, known as $\mathbb{F}_p$ to mathematicians, $GF(p)$ to computer scientists.

Reformulation of Fermat's little theorem

Using the existence of multiplicative inverses in $\mathbb{F}_p$, we can rewrite Fermat's little theorem as follows:

Either p divides a and all its powers or

$$a^{p-1} \equiv 1 \mod p.$$

Extended Euclidean algorithm

Algorithmically, our existence proof is not practicable.

Instead, use the extended Euclidean algorithm for determining the greatest common divisor. This yields a constructive proof of a stronger result:

Bezout's Lemma says that given a and n, there exist b and k such that

$$ab + kn = \gcd(a, n).$$

Example: $n = 7$ and $a = 3$.

Basics of RSA encryption

Secret: p and q, two large prime numbers.

Public: the product $n = pq$, to be used as modulus.

Secret: the product $(p-1)(q-1)$.

Public: the encryption key e, which is chosen to be coprime to $(p-1)(q-1)$.

Secret: the decryption key $d = e^{-1}$ modulo $(p-1)(q-1)$.

To encrypt a message m, form the eth power $c = m^e$ modulo n. Then send the encrypted message c.

The receipient decodes the message by taking the dth power of c,

$$c^d \equiv m \mod n.$$

Why does it work?

Theorem:

$$(c^d) = m^{ed} \equiv m \qquad \text{mod } n.$$

Proof: We need to show that

$$p \text{ divides } m^{ed} - m \qquad \text{and} \qquad q \text{ divides } m^{ed} - m.$$

By Bezout's Lemma there exists $k \in \mathbb{N}$ such that

$$ed = 1 + k(p-1)(q-1)$$

Hence

$$\begin{aligned} m^{ed} &\equiv \left(m^{k(q-1)}\right)^{p-1} \times m \\ &\equiv m \end{aligned}$$

by Fermat's little theorem.