The Rank Nullity Theorem and Singular Value Decomposition

A contribution to Linear Algebra

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The four fundamental subspaces

Let A be an $n \times m$-matrix with entries in a field $\mathbb{F}$. Associated to A we have four linear subspaces:

1. The kernel or null-space of A, denoted $N(A)$, consists of all elements $\vec{x} \in \mathbb{F}^m$ such that $A\vec{x} = \vec{0}$.
2. The column space of A, denoted $C(A)$, is the subspace of $\mathbb{F}^n$ spanned by the column vectors $\vec{c}_1, \ldots \vec{c}_m$.
3. The row space of A, denoted $C(A^T)$ is the subspace of $\mathbb{F}^m$ spanned by the transposes of the row vectors $\vec{r}_1, \ldots, \vec{r}_n$.
4. The null-space of A^T, denoted $N(A^T)$, consists of all $\vec{y} \in \mathbb{F}^n$ such that $A^T\vec{y} = \vec{0}$.

These spaces are invariant under elementary row operations. If the matrix A is in row-echelon form, it is easy to read off bases and dimensions of these spaces. Try some examples. What do you notice?

Null space and row space

The null space and the row space are orthogonal complements of each other (in the ambient vector space $\mathbb{F}^m$). This means that $N(A)$ consists exactly of all the vectors in $\mathbb{F}^m$ that are orthogonal to $C(A^T)$, and vice versa: a vector is in $C(A^T)$ if and only it is orthogonal to every vector in the null-space $N(A)$.

Proof:

Column Space and Null Space of the Transpose

The argument on the previous slide goes through to show that the column space is the orthogonal complement of $N(A^T)$ inside $\mathbb{R}^n$.

The column space agrees with the image of the linear transformation given by (left-)muliplication with A. This is the sub-space of all vectors $\vec{b} \in \mathbb{F}^n$ such that the system of equations $A\vec{x} = \vec{b}$ has at least one solution.

Example:

Isomorphism Between Row and Column Space

Theorem: Multiplication with A maps the row space isomorphically to the column space

$$\begin{aligned} C(A^T) &\xrightarrow{\cong} C(A) \\ \vec{r} &\longmapsto A\vec{r}. \end{aligned}$$

Proof: Let $\vec{b} = A\vec{x}$. We have

$$\vec{x} = \vec{n} + \vec{r},$$

where

$$\vec{n} = proj_{N(A)}\vec{x} \quad \text{and} \quad \vec{r} = proj_{C(A^T)}\vec{x}$$

are the orthogonal projections of x to the null and the row space. Hence

$$\vec{b} = A\vec{x} = A\vec{n} + A\vec{r} = A\vec{r},$$

proving surjectivity. If $\vec{b} = \vec{0}$ then $\vec{x} = \vec{n}$ is already in the null-space, hence $\vec{r} = 0$, showing injectivity. □

Rank and Nullity

The theorem implies that the dimensions of row and column space of any given matrix A are equal. This dimension is called the rank of A, denoted

$$rank(A) = dim(C(A)) = dim(C(A^T)).$$

The dimension of the null-space is called nullity. Because of the orthogonal decompositions, we have

$$\begin{aligned} dim(N(A)) &= m - rank(A) \\ dim(N(A^T)) &= n - rank(A). \end{aligned}$$

These equations are known as the rank nullity theorem.
If you want to compute the rank and the nulltity of a matrix, just bring it into row echelon form and read off the answer.