Singular Value Decomposition

A contribution to Linear Algebra

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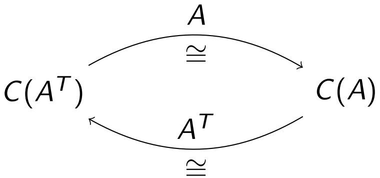
The geometry of the Transpose

Let A be an $n \times m$-matrix with entries in $\mathbb{R}$ and rank r.

We have seen in the lecture on the Rank-Nullity Theorem that multiplication with A defines an isomorphism from the row space to the column space of A.

Applying the same result to A^T, we see that multiplication with A^T defines and isomorphism from the columns space $C(A)$ back to the row space of A.



These two isomorphisms are not inverse to each other.

The Symmetric Square Matrices AA^T and A^TA

Let $\vec{c}_j$ be the jth column of A, and let $\vec{r}_i$ be the ith column of A^T.

The jth column of A^TA is $A^T\vec{c}_j$.

The jth row of A^TA is $(A^T\vec{c}_j)^T$.

The ith column of AA^T is $A\vec{r}_i$.

The ith row of AA^T is $(A\vec{r}_i)^T$.

The $m \times m$ matrix A^TA and the $n \times n$ matrix AA^T are symmetric.

Relationship to the four fundamental subspaces

The column space of A agrees with the row/column space of AA^T.

The row space of A agrees with the row/column space of A^TA.

The null-space of A is equal to the null-space of A^TA.

The null-space of A^T is equal to the null space of AA^T.

The Matrices $A^T A$ and AA^T are Positive Semi-Definite

We have

$$(\vec{v})^T (A^T A)\vec{v} = \langle A\vec{v}, A\vec{v}\rangle \geq 0$$

with equality exactly for the $\vec{v}$ in the null-space of A;

$$(\vec{v})^T (AA^T)\vec{v} = \langle A^T\vec{v}, A^T\vec{v}\rangle \geq 0$$

with equality exactly for the $\vec{v}$ in the null-space of A^T.

In particular all eigenvalues of $A^T A$ and AA^T are non-negative real numbers.

Applying the Spectral Theorem to $A^T A$

Applying the spectral theorem to $A^T A$, we obtain an orthonormal basis $\{\vec{v}_1, \ldots \vec{v}_m\}$ of eigenvectors.

Write $\lambda_i \in \mathbb{R}$ for the eigenvalue for $\vec{v}_i$ and order the basis in such a manner that the corresponding eigenvalues decrease in magnitute, i.e.,

$$\lambda_1 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = \lambda_m = 0.$$

Then the first r eigenvectors form a basis for the row space of A.

The last $m - r$ eigenvectors with eigenvalue zero form a basis for the null-space of A.

Compatible Bases for Row and Column Space

If $\vec{v}$ is an eigenvector for A^TA with eigenvalue $\lambda > 0$ then $A\vec{v}$ is an eigenvector for AA^T for the same eigenvalue.

If $\vec{v} \perp \vec{w}$ are eigenvectors for A^TA, then $A\vec{v} \perp A\vec{w}$.

Set

$$\vec{u}_j = \sqrt{\lambda}_j \, A\vec{v}_j \qquad \text{for } 1 \leq i \leq r.$$

Then

$$\vec{v}_j = \sqrt{\lambda}_j \, A^T \vec{u}_j \qquad \text{for } 1 \leq i \leq r.$$

A Matching Set of Eigenvectors for AA^T

Choose an orthonormal basis $\{\vec{u}_{r+1}, \ldots, \vec{u}_n\}$ of the null space of the transpose, $N(A^T)$.

Then $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is a spectral basis for AA^T, with the first r vectors spanning the column space of A.

Singular Value Decomposition

The singular value decomposition is the matrix identity

$$AV = U\Sigma$$

where V has column vectors $\vec{v}_1, \ldots \vec{v}_m$, and U has column vectors $\vec{u}_1 \ldots, \vec{u}_n$, while Σ is the $n \times m$ block matrix

$$\Sigma = \begin{bmatrix} Diag(\sqrt{\lambda}_1, \ldots \sqrt{\lambda}_r) & 0 \\ 0 & 0 \end{bmatrix}$$

Further reading: Strang Essay