Areas of parallelograms and triangles

This set of slides still doesn't have any pictures

Triangles are half parallelograms, so it is enough to focus on parallelograms.

Let P be the parallelogram spanned by $\vec{u}$ and $\vec{v}$.

In 2 dimensions:

$$\text{oriented area(P)} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1.$$

Why? Check the checklist: the left-hand side satisfies the Determinant Properties 1-4. (This was our motivation for postulating these properties.)

In the current notes, $\text{area} = |\text{oriented area}|$.

In 3 dimensions

$$\text{area}(\text{P}) = \|\vec{u} \times \vec{v}\|$$

Why? Since $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$, the volume of the parallelepiped formed by $\vec{u}$, $\vec{v}$ and $\vec{u} \times \vec{v}$ equals $\|\vec{u} \times \vec{v}\|$ times area(P). That volume also equals

$$det(\vec{u} \times \vec{v}, \vec{u}, \vec{v}) = (\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v})$$

(easy, see slide on "scalar triple product").

Corollary The distance from a point $\vec{v}$ to a line $\ell = span(\vec{u})$ in $\mathbb{R}^3$ is given by the formula

$$\text{distance}\,(\vec{v}, \ell) \quad = \quad \frac{\|\vec{u} \times \vec{v}\|}{\|\vec{u}\|}.$$

Always

$$\text{area(P)} \quad = \quad \sqrt{\|\vec{u}\|^2\|\vec{v}\|^2 - \langle \vec{u}, \vec{v}\rangle^2},$$

and

$$\text{distance}\,(\vec{v}, \ell) \quad = \quad \frac{\text{area(P)}}{\|\vec{u}\|}.$$

Why? If $\theta = \sphericalangle(\vec{u}, \vec{v})$, then the area equals

$$\text{base} \cdot \text{height} = \|\vec{u}\| \cdot \text{distance}\,(\vec{v}, \ell) = \|\vec{u}\|\|\vec{v}\| \sin\theta.$$

The first formula then follows from

$$\sin^2\theta + \cos^2\theta = 1.$$