Bases and Dimension

A contribution to Linear Algebra by Nora Ganter

Standard Basis of $\mathbb{F}^n$

The standard basis of the vector space $\mathbb{F}^n$ is the set $\mathcal{S}$ consisting of the vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Every vector of $\mathbb{F}^n$ can be expressed, in a unique manner, as a linear combination of the standard basis vectors:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n.$$

Coordinate Frames and bases

Let V be a vector space over $\mathbb{F}$, let

$$T \colon \mathbb{F}^n \longrightarrow V.$$

be a linear transformation. Recall that T is completely determined by the ordered set consisting of the vectors

$$\begin{aligned} \vec{b}_1 &= T(\vec{e}_1) \\ &\vdots \\ \vec{b}_n &= T(\vec{e}_n). \end{aligned}$$

Definition: If T is bijective, then the set

$$\mathcal{B} = \{\vec{b}_i\}_{i=1}^n$$

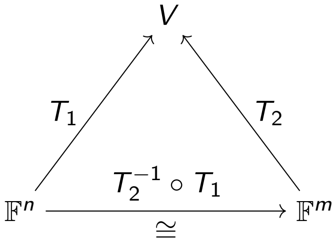
is called a basis of V, and T is called a coordinate frame for V.

Change of Coordinate Frame, Dimension

Given two coordinate frames

$$T_1 \colon \mathbb{F}^n \xrightarrow{\cong} V \quad \text{and} \quad T_2 \colon \mathbb{F}^m \xrightarrow{\cong} V$$

for the same vector space V, we have the change of coordinate matrix describing the linear transformation $T_2^{-1} \circ T_1$.



Since invertible matrices are square, it follows that $n = m$. This number is called the dimension of V.

Span and Spanning Sets

Definition: The image of T is also called the span of $\mathcal{B}$, denoted $span(\mathcal{B})$.

If $T\colon \mathbb{F}^n \longrightarrow V$ is surjective (but not necessarily injective) then $span(\mathcal{B}) = V$, and $\mathcal{B}$ is called a spanning set for V.

Elements in the span of $\mathcal{B}$ are exactly those elements of V that can be written as a linear combination of vectors in $\mathcal{B}$. Note that such a presentation might not be unique.

Show that $span(\mathcal{B})$ is a linear subspace of V:

Examples

1. column space 2. an example with polynomials 3. a picture with some questions.

Linear Independence

Definition: If T is injective, but not necessarily surjective, then $\mathcal{B}$ is called linearly independent.
T is injective means

$$T(\vec{x}) = T(\vec{y}) \quad \Longrightarrow \quad \vec{x} = \vec{y}.$$

Because of the linearity of T, this is equivalent to the triviality of the null-space of T, i.e., to

$$T(\vec{x}) = \vec{0} \quad \Longrightarrow \quad \vec{x} = \vec{0}.$$

Proof:

In terms of Linear Combinations

The vectors $\vec{b}_1, \ldots \vec{b}_n$ are linearly independent if and only if

A subset of V that is not linearly independent is called linearly dependent.

Examples and Pictures

Ordered versus unordered bases

Let V be a vectors space over a field $\mathbb{F}$. A basis is a linearly independent spanning set $\mathcal{B}$ of V.
If a basis has finitely many elements and we have specified an ordering on these elements,

$$\mathcal{B} = \{\vec{b}_1, \ldots \vec{b}_n\}$$

then these data give rise to a coordinate frame

$$\begin{aligned} T_{\mathcal{B}} \colon \mathbb{F}^n &\xrightarrow{\cong} V \\ \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} &\longmapsto a_1\vec{b}_1 + \cdots + a_n\vec{b}_n. \end{aligned}$$

Example

The linear transformations T_1 and T_2 described by the matrices

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix}$$

both give the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

of $\mathbb{R}^3$, but with different orderings of the elements.

Note: Often the term "change of basis" is used synonymously with "change of coordinate frames". Implicit in this phrase is the assumtion that an ordering for both bases has been chosen.

Infinite dimensional case

Above we have restricted ourselves to the case where n is a finite number. This makes it a little easier to formulate the theory and will be sufficient for the purposes of this beginners class.

In practice, you have already encountered examples of infinite dimensional vector spaces, such as the space of polynomials with coefficients in $\mathbb{F}$ (countably infinite dimensional over $\mathbb{F}$) or the real numbers, viewed as a vector space over the rationals (uncountably infinite dimensional).

If you are dealing with infinite dimensional vector spaces in real world applications, you will often times find yourself working with finite dimensional approximations. To decide whether such an approximation is any good, you need a notion of distance.