

The Determinant of a Square Matrix

[AR 2.1–2.3]

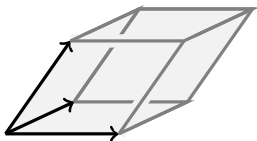
The row vectors of an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

are

$$(a_{11}, a_{12}, \dots, a_{1n}) \quad \dots \quad (a_{n1}, a_{n2}, \dots, a_{nn}).$$

Goal: To define and to calculate the *oriented volume* of the *parallelepiped* (higher dimensional parallelogram) defined by the row vectors of A .



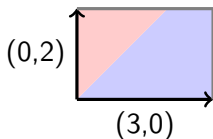
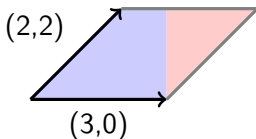
Once defined, we will call this the *determinant* of A , denoted $\det(A)$ or simply $|A|$.

Some 2-dimensional examples

The parallelograms of the row vectors of

$$A = \begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix} \quad \text{and}$$

$$B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$



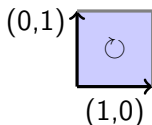
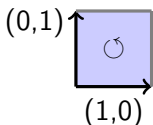
have the same surface area, namely $3 \cdot 2 = 6$.

Examples for orientation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

versus

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



counter clock-wise orientation, the oriented surface area equals 1

clock-wise orientation, the oriented surface area equals -1

In 3 dimensions:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

gives a right-handed system with oriented volume 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

gives a left-handed system with oriented volume -1

Postulate

We want the assignment

$$A \mapsto \det(A)$$

to have the following *properties*:

1. If B is obtained from A by *swapping* two rows of A , then $\det(B) = -\det(A)$
2. If B is obtained from A by *multiplying a row* of A by the scalar α , then $\det(B) = \alpha \det(A)$
3. If B is obtained from A by replacing a row of A by itself *plus a multiple of another row*, then $\det(B) = \det(A)$
4. The determinant of any *identity matrix* is equal to **1**.

We yet need to convince ourselves that such a function $A \mapsto \det(A)$ actually exists and that it is completely characterized by these properties. Then we will be allowed to use this slide as definition of $\det(A)$.

Computer lab 3 will cover determinants as volume and the effect of row operations in detail.

How to calculate determinants

Using the properties of the previous slide, $\det(A)$ is calculated as follows: *If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A .*

Examples

$$\begin{bmatrix} 2 & -1 & 9 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & -3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

For a general matrix, do Gauss elimination to bring A into row echelon form, keeping track of what each step does to the determinant (it is best to avoid the 2nd elementary row operation).

This is by far the sturdiest method for calculating determinants. The other methods are almost guaranteed to produce sign mistakes!

Examples

$$\text{Calculate } \begin{vmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{vmatrix} \text{ and } \begin{vmatrix} 2 & -4 & 1 \\ 3 & -6 & 3 \\ 2 & 1 & 4 \end{vmatrix}.$$

$$\text{Calculate } \begin{vmatrix} 1 & -2 & 0 & 1 \\ 3 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 2 \end{vmatrix} \text{ and } \begin{vmatrix} 2 & -10 & 92 & -117 \\ 0 & 3 & 28 & -31 \\ 0 & 0 & -1 & 27 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

The determinant of a 2×2 matrix is $ad - bc$.

Proof:

$$\text{if } a \neq 0 \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - \frac{cb}{a} \end{vmatrix} = ad - bc$$

$$\text{and if } a = 0 \quad \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = - \begin{vmatrix} c & d \\ 0 & b \end{vmatrix} = -bc.$$

Leibniz formula

A powerful theorem by Leibniz, which we will not pretend to prove, says that

$$\det(A) = \sum_{\sigma \in S_n} \prod_{i=1}^n \operatorname{sgn}(\sigma) a_{i,\sigma(i)}$$

satisfies the properties we postulated above.

You will not be asked to calculate anything using Leibniz' formula. To explain the symbols: the sum ranges over all shuffles (permutations) of $\{1, \dots, n\}$ and $\operatorname{sgn}(\sigma) = \pm 1$ depends on the number of flips in the shuffle σ . (Even that $\operatorname{sgn}(\sigma)$ is well-defined is a non-trivial fact.)

We have already seen how to calculate the determinant, but it was not at all clear why two different ways to do Gauss elimination would yield the same result. This is one great thing about Leibniz' definition: it is given directly in terms of A .

Other important properties of the determinant

5. $\det(A^T) = \det(A)$
6. $\det(AB) = \det(A) \det(B)$
7. if A is an $n \times n$ matrix then $\det(\alpha A) = \alpha^n \det(A)$
8. If A has a row (or column) of zeros, then $\det(A) = 0$
9. If A has a row (or column) which is a scalar multiple of another row (or column) then $\det(A) = 0$
10. A is singular if and only if $\det(A) = 0$
11. A is invertible if and only if $\det(A) \neq 0$, and in this case $\det(A^{-1}) = \frac{1}{\det(A)}$
12. If $A = \left[\begin{array}{c|c} C & * \\ \hline 0 & D \end{array} \right]$ with C, D square, then $\det(A) = \det(C) \det(D)$

Additivity in the rows (Property 3')

Determinants are *not well behaved with respect to matrix addition*.

Example

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \quad A + B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

have determinants

Instead, we have: *If the first row of C equals the sum of the first rows of A and B and all the other entries are identical in A, B and C* then

$$\det(C) = \det(A) + \det(B),$$

and similarly for the i th row (or column).

Example

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} a' & b' \\ c & d \end{bmatrix} \quad C = \begin{bmatrix} a + a' & b + b' \\ c & d \end{bmatrix}$$

Multilinearity

The property of the last slide looks similar to Property 2 (scaling a (single) row with a scalar α multiplies the determinant by the same scalar), and together these two properties are known as the *multilinearity property* of the determinant.

Properties 1,2,3' and 4 are often used to characterize the determinant. This approach is equivalent to our characterization by Properties 1-4.

Idea of proof:

Laplace formula for $\det(A)$, a.k.a. cofactor expansion

The (i, j) -cofactor C_{ij} of an $n \times n$ matrix A is the number

$$C_{ij} = (-1)^{i+j} \det(A(i, j))$$

where $A(i, j)$ is the matrix obtained from A by deleting the i th row and j th column.

To compute the determinant of A , choose any row (or column) of A , multiply each entry in that row (or column) by its cofactor and add the results. So, we obtain the *cofactor expansion along the i th row*:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

and the *cofactor expansion along the j th column*:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Examples

$$\text{If } A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \text{ then } A(2,3) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } C_{23} =$$

$$\text{Calculate } \det \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\text{Calculate } \begin{vmatrix} 1 & -2 & 0 & 1 \\ 3 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 2 \end{vmatrix}$$

Cofactor expansion should be in your toolkit. Usually, Gauss elimination works better, but if you can spot a column (or row) with many zeroes, go for cofactor expansion along it!

How do you remember the sign of the cofactor?

The $(1, 1)$ -cofactor always has sign $+$. Starting from there, imagine walking to the square you want using either horizontal or vertical steps. The appropriate sign will change at each step.

We can visualize this arrangement with the following matrix:

$$\begin{bmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

So, for example, C_{13} is assigned $+$ but C_{32} is assigned $-$