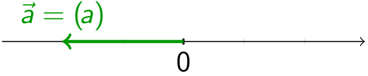
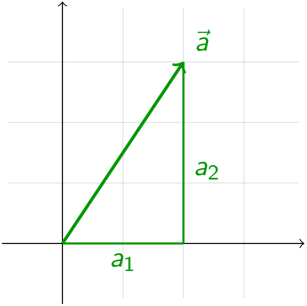
The length of a vector: our geometric intuition

$\mathbb{R}^1$

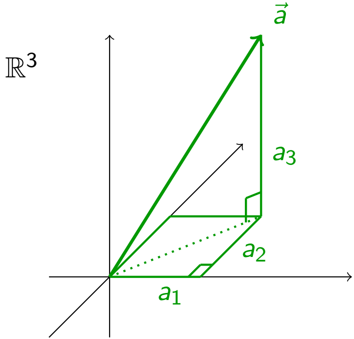


$\|\vec{a}\| = |a|$ absolute value

$\mathbb{R}^2$



$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$ (Pythagoras)



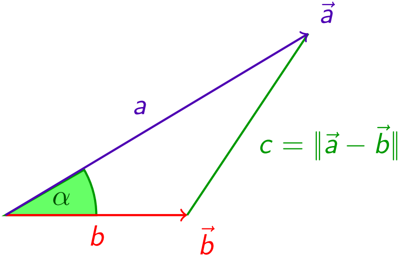
$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

(apply Pythagoras twice)

Formal definition of length: The length of $\vec{a} \in \mathbb{R}^n$ is

$$\|\vec{a}\| = \sqrt{a_1^2 + \cdots + a_n^2} = \sqrt{\langle \vec{a}, \vec{a} \rangle}.$$

Angles between vectors: our geometric intuition



Comparing

$$c^2 \;=\; a^2 + b^2 - 2ab \cdot \cos(\alpha) \qquad \text{(Cosine Theorem)}$$

$$\|\vec{a} - \vec{b}\|^2 \;=\; \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\langle \vec{a}, \vec{b} \rangle \qquad \text{(inner product rules)}$$

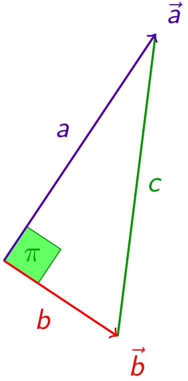
motivates the **formal definition of the angle** $\alpha = \sphericalangle\left(\vec{a}, \vec{b}\right)$ by

$$\cos(\alpha) \overset{def}{=} \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\| \cdot \|\vec{b}\|}, \qquad \alpha \in [0, \pi].$$

Hopefully this works, and the right-hand side is $\leqslant 1$! We will see.

Perpendicular vectors

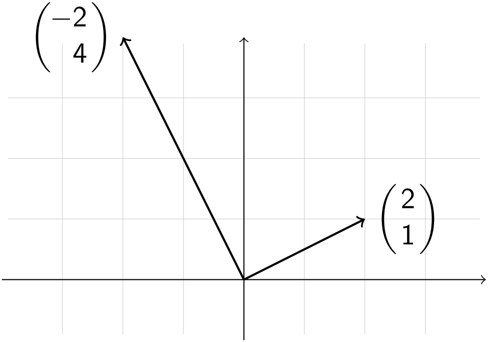
The vectors $\vec{a}$ and $\vec{b}$ are called perpendicular to each other, denoted $\vec{a} \perp \vec{b}$, if $\langle \vec{a}, \vec{b} \rangle = 0$.



So, $\vec{a} \perp \vec{b}$ if and only if Pythagoras' theorem holds: $c^2 = a^2 + b^2$.

Synonyms for "perpendicular" are: orthogonal, or at a right angle, or at a 90° angle, or forming an angle of π, or normal to each other.

Example of two perpendicular vectors in $\mathbb{R}^2$

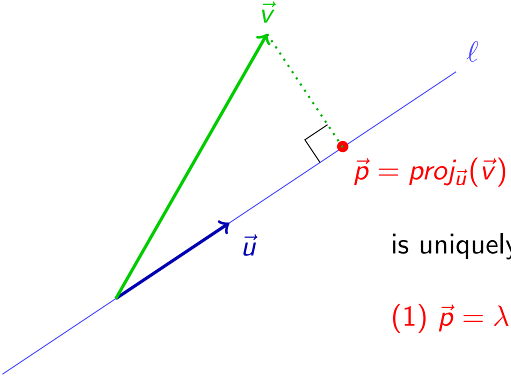


The dot product is

$$(-2, 4)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = -4 + 4 = 0.$$

Orthogonal projections

The projection of $\vec{v}$ onto the line ℓ spanned by $\vec{u} \neq 0$



is uniquely characterized by

(1) $\vec{p} = \lambda \vec{u}$ with $\lambda \in \mathbb{R}$,

(2) $(\vec{v} - \vec{p}) \perp \vec{u}$. Indeed,

$$(\vec{v} - \lambda \vec{u}) \perp \vec{u} \iff \langle \vec{v}, \vec{u} \rangle - \lambda \langle \vec{u}, \vec{u} \rangle = 0 \iff \lambda = \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle}.$$

This yields the **Projection Formula:**

$$proj_{\vec{u}}(\vec{v}) \;=\; \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{u}, \vec{u} \rangle} \cdot \vec{u}.$$

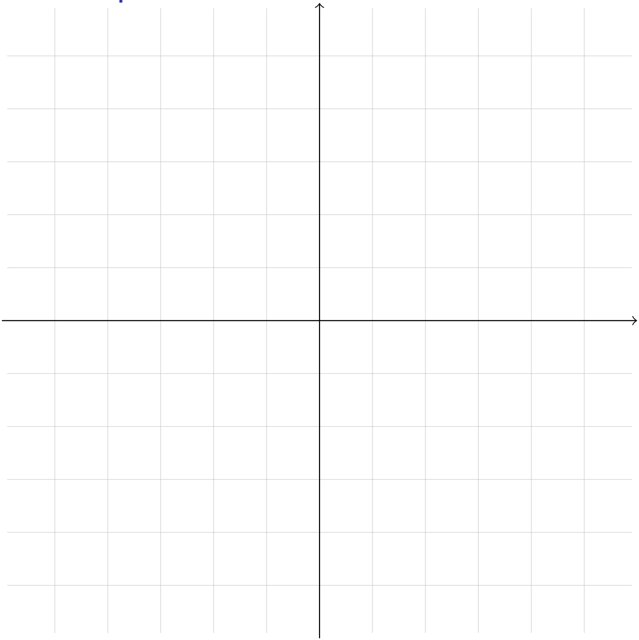
Viewed as a function of $\vec{v}$,

$$proj_{\vec{u}}(\vec{v}) \;=\; \frac{\vec{u}\,\vec{u}^T}{\|\vec{u}\|^2}\,\vec{v}$$

is (left-) multiplication by the **projection matrix**

$$\frac{\vec{u}\,\vec{u}^T}{\vec{u}^T\,\vec{u}}.$$

An example in $\mathbb{R}^2$



$$\vec{u} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$\vec{v} = \begin{pmatrix} 4 \\ -7 \end{pmatrix} \text{ or}$$

$$\vec{v} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Example (continued)

with the same choice of $\vec{u}$, but arbitrary $\vec{v}$

$$
\begin{aligned}
proj_{\begin{pmatrix}-3\\2\end{pmatrix}} \begin{pmatrix}v_1\\v_2\end{pmatrix} &= \\
&= \frac{1}{\quad} \left(v_1 \begin{pmatrix} \\ \end{pmatrix} + v_2 \begin{pmatrix} \\ \end{pmatrix} \right) \\
&= \frac{1}{\quad} \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix}v_1\\v_2\end{pmatrix}.
\end{aligned}
$$

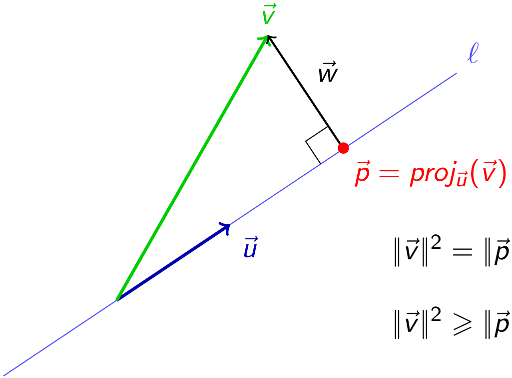
and indeed, the projection matrix is

$$
\frac{\begin{pmatrix}-3\\2\end{pmatrix}(-3,2)}{(-3,2)\begin{pmatrix}-3\\2\end{pmatrix}} = \frac{1}{\quad} \begin{pmatrix} & \\ & \end{pmatrix}
$$

Draw the nullspace, row-space and column space of this projection matrix, and discuss their meaning!

Cauchy-Schwarz inequality

Projection makes the vector shorter: this is an easy consequence of Pythagoras' theorem.



$$\|\vec{v}\|^2 = \|\vec{p}\|^2 + \|\vec{w}\|^2 \quad \Longrightarrow$$

$$\|\vec{v}\|^2 \geqslant \|\vec{p}\|^2 \quad \Longleftrightarrow$$

$$\|\vec{v}\| \geqslant \|\vec{p}\| = \frac{|\langle \vec{v}, \vec{u} \rangle|}{\|\vec{u}\|} \quad \Longleftrightarrow$$

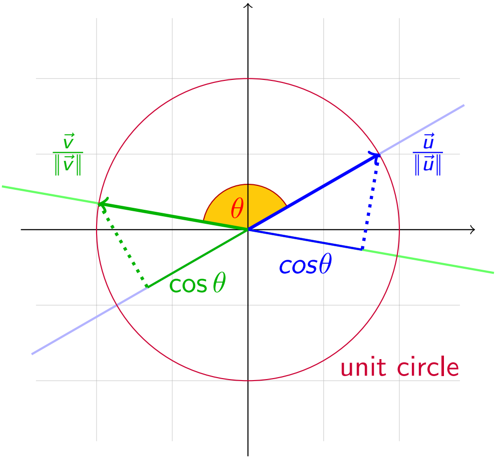
$$\|\vec{v}\| \cdot \|\vec{u}\| \geqslant |\langle \vec{v}, \vec{u} \rangle| \quad \text{(Cauchy-Schwarz).}$$

Equality holds if and only if $\vec{u}$ and $\vec{v}$ lie in the same line.

Angles between vectors

Now, using Cauchy-Schwarz, we may define angles the way we hoped to!

A second way to visualize the definition of angle:



In both of these triangles, the hypothenuse has length 1. Hence

$$proj_{\frac{\vec{u}}{\|\vec{u}\|}}\left(\frac{\vec{v}}{\|\vec{v}\|}\right) = \cos\theta\,\frac{\vec{u}}{\|\vec{u}\|},$$

and similarly for the blue right triangle. In other words,

$$\cos\theta \;=\; \left\langle \frac{\vec{v}}{\|\vec{v}\|}, \frac{\vec{u}}{\|\vec{u}\|} \right\rangle.$$

(In the picture, $\cos\theta$ is negative.)