Matrix Multiplication and Matrix Inverses

Nora Ganter, University of Melbourne

Multiplying a Single Row Matrix with a Vector

The dot product

$$\vec{a}^t \circ \vec{x} \;=\; \vec{a}^t \vec{x} \;=\; [\vec{a} \bullet \vec{x}]$$

(dot product).

Examples:

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [1 \cdot 3 + 2 \cdot 4] = [11]$$

$$\begin{bmatrix} 7 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [4] \qquad \text{means} \qquad 7x + 3y + z = 4$$

Formula:

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [a_1 x_1 + a_2 x_2 + \cdots + a_n x_n]$$

Multiplying a Matrix with a Vector – Row by Row

$$A\vec{x} = \begin{bmatrix} \vec{a}_1^t \\ \hline \vec{a}_2^t \\ \hline \cdots \\ \hline \vec{a}_m^t \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{a}_1 \bullet \vec{x} \\ \hline \vec{a}_2 \bullet \vec{x} \\ \hline \cdots \\ \hline \vec{a}_m \bullet \vec{x} \end{bmatrix}$$

Examples:

$$\begin{bmatrix} 1 & 2 \\ \hline 5 & 0 \\ \hline -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ \hline 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 2 \cdot 4 \\ \hline 5 \cdot 3 + 0 \cdot 4 \\ \hline -1 \cdot 3 + 1 \cdot 4 \end{bmatrix} = \begin{bmatrix} 11 \\ \hline 15 \\ \hline 1 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 3 & 1 \\ \hline 0 & 8 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ \hline 1 \end{bmatrix}$$

means

$$7x + 3y + z = 4$$

and

$$8y + z = 1$$

Multiplying a Matrix with a Vector – Column Point of View

Example

$$\left[\begin{array}{c|c|c} 7 & 3 & 1 \\ 0 & 8 & 1 \end{array}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 7 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 8 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

In general

$$\left[\begin{array}{c|c|c|c} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \cdot \vec{a}_1 + x_2 \cdot \vec{a}_2 + \dots + x_n \cdot \vec{a}_n$$

The Linear Transformation

Write

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

for the standard basis. Multiplication with A is a linear transformation, and the columns of A are the values of the $\vec{e}_i$.

$$A\vec{e}_1 = \vec{a}_1, \qquad A\vec{e}_2 = \vec{a}_2, \qquad \dots \qquad A\vec{e}_n = \vec{a}_n$$

The transformation is determined by these values and linearity

$$A\left(x_1 \cdot \vec{e}_1 + x_2 \cdot \vec{e}_2 + \dots + x_n \cdot \vec{e}_n\right) \;=\; x_1 \cdot \vec{a}_1 + x_2 \cdot \vec{a}_2 + \dots + x_n \cdot \vec{a}_n$$

Summary

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \cdot \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Multiplying matrices with each other

Composition of linear transformations

$$AB = A \left[\begin{array}{c|c|c|c} \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_n \end{array} \right] = \left[\begin{array}{c|c|c|c} A\vec{b}_1 & A\vec{b}_2 & \dots & A\vec{b}_n \end{array} \right]$$

The jth column is $A(B(\vec{e}_j))$.

$$\left[\begin{array}{c} \vec{a}_1^t \\ \hline \vec{a}_2^t \\ \hline \dots \\ \hline \vec{a}_m^t \end{array} \right] \left[\begin{array}{c|c|c|c} \vec{b}_1 & \vec{b}_2 & \dots & \vec{b}_n \end{array} \right] = \begin{bmatrix} \vec{a}_1^t \vec{b}_1 & \vec{a}_1^t \vec{b}_2 & \dots & \vec{a}_1^t \vec{b}_n \\ \vec{a}_2^t \vec{b}_1 & \vec{a}_2^t \vec{b}_2 & \dots & \vec{a}_2^t \vec{b}_n \\ & \dots & & \\ \vec{a}_m^t \vec{b}_1 & \vec{a}_m^t \vec{b}_2 & \dots & \vec{a}_m^t \vec{b}_n \end{bmatrix}$$

The entry in row i and column j is the dot product of the ith row of A with the jth column of B.

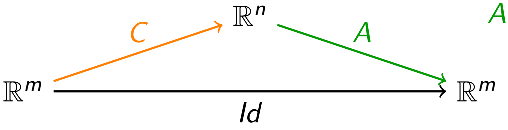
The Identity Matrix I

Right Inverses

A right inverse of a given matrix A is a matrix C characterized by

$$AC = I.$$

As linear transformations:



$$A(C(\vec{x})) = \vec{x}$$

This is satisfied if and only if the columns of

$$C = \left[\begin{array}{c|c|c|c} \vec{c}_1 & \vec{c}_2 & \dots & \vec{c}_n \end{array}\right]$$

are solutions of

$$A\vec{c}_i = \vec{e}_i.$$

Proof

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$C\vec{x} =$$

$$AC\vec{x} =$$

The inverse of a square matrix

If A is an $n \times n$ matrix, then one can show

$$AC = I \quad \iff \quad CA = I.$$

If such a C exists, it is uniquely determined and called the inverse of A, denoted A^{-1}.

So, A has a inverse if and only if we can solve the n systems of equations

$$A\vec{c_i} = \vec{e_i},$$

This is done simultaneously using the Gauß-Jordan Algorithm.

Inverse of a Square Matrix, Algorithmic Point of View

Goal: determine whether A possesses an inverse and if so, find the inverse.

Starting Point:

$$\left[\begin{array}{c|c|c|c} & 1 & 0 & 0 \\ A & 0 & \ddots & 0 \\ & 0 & 0 & 1 \end{array}\right].$$

Apply Gauß-Jordan Algorithm, treating all columns at once, aiming to arrive at

$$\left[\begin{array}{c|c|c|c|c|c} 1 & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & \vec{c}_1 & \dots & \vec{c}_n \\ 0 & \cdots & 1 & & & \end{array}\right].$$

If this fails, then A is not invertible. Such an A is called singular. Otherwise, A is called regular, and A^{-1} is given by the RHS.