Special relativity theory

Indigo sits in her spaceship, dreaming out of the window and sees her friend Ruby pass in another spaceship in a straight line at constant velocity $v > 0$.

To Ruby, this looks a bit different, from her perspective, she is stationary, and Indigo is the one who is moving in a straight line with constant velocity $-v$.

Since they cannot figure out who is right, they agree to disagree. This means that they will need to find a way to translate from Ruby's bookkeeping to Indie's and back.

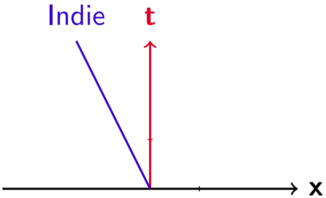
For simplicity, we assume that the two spaceships pass each other in the same spot at the same time.

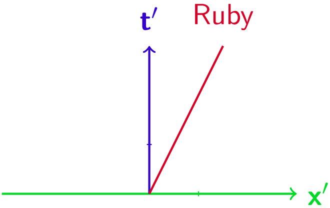
You might even have been in a similar situation, sitting in the train and looking out of the window to see another train, seemingly moving, but maybe it is you?

Note that only one space dimension is relevant to the question who is moving, namely the direction of (perceived) movement.

The girls draw the following pictures of the world

They agree to make the point where they met the origin.





Ruby's point of view:
red t-axis (time),
black x-axis (space direction of movement),
blue line

$$x = -vt$$

describing Indie's movement.

Indigo's point of view:
blue t'-axis (time),
green x'-axis (space direction of movement),
red line

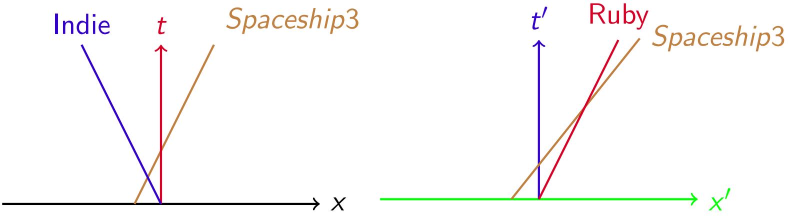
$$x' = vt'$$

describing Ruby's movement.

Change of reference frame

Each girl would like to know how the world looks like from her friend's perspective.

Observation: The girls start to look at other objects. They cannot agree whether they are moving or not or how fast, but they always agree on whether an object is moving at constant speed (rectilinear motion):



They postulate: The change of reference frame is a linear transformation.

So, the girls are looking for a 2×2 matrix $A = A_{Indie,Ruby}$ that will translate Ruby's picture into Indigo's:

$$\begin{bmatrix} t' \\ x' \end{bmatrix} = A \begin{bmatrix} t \\ x \end{bmatrix}.$$

Careful: the physics convention is to put the time coordinate first, although the pictures might make you expect otherwise.

The velocity v is known. The point that has coordinates $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in Ruby's picture can be found somewhere on the red Ruby line $span(\begin{bmatrix} 1 \\ v \end{bmatrix})$ in Indigo's picture. So, A is of the form

$$A = \begin{bmatrix} a & b \\ & d \end{bmatrix}$$

We note that $a > 0$ (postulate that time does not get reversed)

The symmetry

The two pictures in the beginning are symmetric around the line $span(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$. The reflection of $\mathbb{R}^2$ at this line has matrix representation

$$R = \begin{bmatrix} & \\ & \end{bmatrix}$$

The inverse of R has matrix representation

$$R^{-1} = \begin{bmatrix} & \\ & \end{bmatrix}$$

The inverse transformation

We also have the matrix

$$A_{Ruby,Indie} = A^{-1}_{Indie,Ruby} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

that translates from Indigo's picture to back Ruby's.

Exploiting the symmetry, the two girls postulate

$$A_{Ruby,Indie} = R\ A_{Indie,Ruby}\ R^{-1}.$$

Calculations

From this equation, they can deduce that $det(A) = 1$ and express d in terms of a.

The speed of light

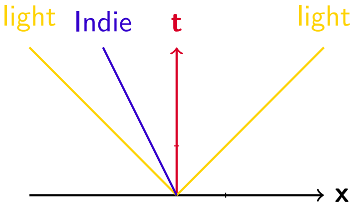
To settle their argument about who is moving and who is stationary, the girls have a great idea. The moment and place where they met, there was a light-flash. Indigo tells Ruby: "To me, you appear to be going at half the speed of light. If that is the case, the light should seem slower to you in the direction you are going and faster in the other direction."

Both girls have a look at the speed of light. Both find that the light is equally fast in both directions. Both find the same speed, namely

$$c = 299792458$$

metres per second.

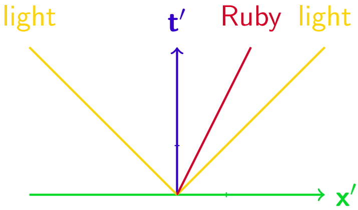
In orther words, they observe that the light rays are eigenspaces of A.



Ruby's point of view:
red t-axis (time), x-axis
(space), blue line

$$x = -vt$$

describing Indie's movement,
yellow light rays.



Indigo's point of view:
blue t'-axis (time), x'-axis
(space), red line

$$x' = vt'$$

describing Ruby' movement,
yellow light rays.

Calculations

We have

$$A \begin{bmatrix} 1 \\ c \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ c \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ -c \end{bmatrix} = \beta \begin{bmatrix} 1 \\ -c \end{bmatrix},$$

where the scalars α and β are the eigenvalues. Use this to express b in terms of a, the velocity v and the speed of light c.

The Lorentz transformation

Let $L \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation given by left-multiplication with A. It is called the Lorentz transformation for v. In terms of the basis

$$\mathcal{L} := \left(\begin{bmatrix} 1 \\ c \end{bmatrix}, \begin{bmatrix} 1 \\ -c \end{bmatrix} \right)$$

the Lorentz transformation L has matrix representation

$$[L]_{\mathcal{L},\mathcal{L}} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

(Express the answer in terms of the scalars α and β.)
It follows that

$$\alpha \cdot \beta =$$

The standard matrix of the Lorentz transformation

We are now able to express the first entry a of A in terms of the velocity v and the speed of light c.

We have shown

$$A = \begin{bmatrix} \\ \end{bmatrix}.$$

Just for fun

To double check their result, the girls decide to consider the base change between the basis of light vectors $\mathcal{L}$ and the standard basis $\mathcal{S}$ of $\mathbb{R}^2$. The change of coordinate matrices

$$[P]_{\mathcal{S},\mathcal{L}} = \begin{bmatrix} & \\ & \end{bmatrix}$$

and

$$[P]_{\mathcal{L},\mathcal{S}} = \begin{bmatrix} & \\ & \end{bmatrix},$$

and indeed,

$$[P]_{\mathcal{L},\mathcal{S}}\ A\ [P]_{\mathcal{S},\mathcal{L}} = \begin{bmatrix} & \\ & \end{bmatrix} = [L]_{\mathcal{L},\mathcal{L}}.$$