

Special relativity theory

Indigo sits in her spaceship, dreaming out of the window and sees her friend Ruby pass in another spaceship in a straight line at constant velocity $v > 0$.

To Ruby, this looks a bit different, from her perspective, she is stationary, and Indigo is the one who is moving in a straight line with constant velocity $-v$.

Since they cannot figure out who is right, they agree to disagree. This means that they will need to find a way to translate from Ruby's bookkeeping to Indie's and back.

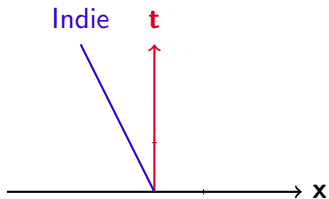
For simplicity, we assume that the two spaceships pass each other in the same spot at a given point of time.

The girls own identical clocks and space measures.

Note that only one space dimension is relevant to the question who is moving, namely the direction of (perceived) movement.

The girls draw the following pictures of the world

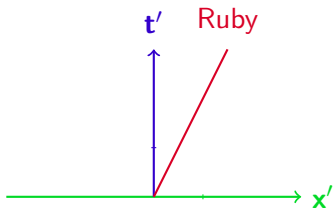
They agree to make the point where they met the origin.



Ruby's point of view:
red t -axis (time),
black x -axis (space direction
of movement),
blue line

$$x = -vt$$

describing Indie's movement.



Indigo's point of view:
blue t' -axis (time),
green x' -axis (space direction
of movement),
red line

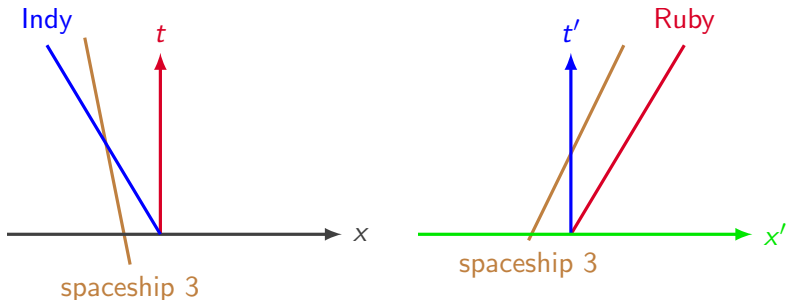
$$x' = vt'$$

describing Ruby's movement.

Change of reference frame

Each girl would like to know how the world looks like from her friend's perspective.

Observation: The girls start to look at other objects. They cannot agree whether they are moving or not or how fast, but they always agree on whether or not an object is moving at constant speed (**rectilinear motion**):



They postulate: Lines go to lines.

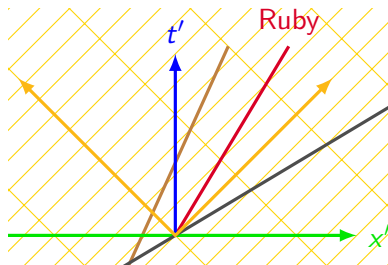
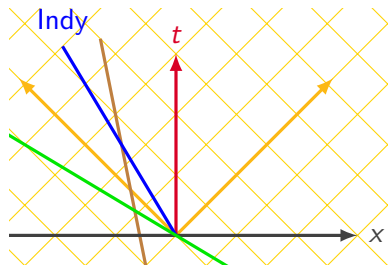
Algebraically, this means

Compatibility with **sums** and **multiplication by scalars**:

$$\begin{array}{ccc} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \longleftrightarrow & \begin{bmatrix} a \\ va \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \longleftrightarrow & \begin{bmatrix} b \\ d \end{bmatrix} \\ \begin{bmatrix} a \\ -va \end{bmatrix} & \longleftrightarrow & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ (observe symmetry!)} \\ \begin{bmatrix} b \\ -d \end{bmatrix} & \longleftrightarrow & \begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{ (postulate symmetry!)} \\ \begin{bmatrix} t \\ x \end{bmatrix} & \longleftrightarrow & t \begin{bmatrix} a \\ va \end{bmatrix} + x \begin{bmatrix} b \\ d \end{bmatrix} \end{array}$$

Here v is the relative velocity and a and b and d are **unknown**.

The speed of light is $c = 299792458$ for both girls!



$$\begin{matrix} \begin{bmatrix} 1 \\ c \end{bmatrix} & \longleftrightarrow & \lambda \cdot \begin{bmatrix} 1 \\ c \end{bmatrix} \\ \lambda \cdot \begin{bmatrix} 1 \\ -c \end{bmatrix} & \longleftrightarrow & \begin{bmatrix} 1 \\ -c \end{bmatrix} \quad (\text{symmetry}) \end{matrix}$$

light rays (through the origin) are eigenspaces.

The Calculation

We know

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \cdot \begin{bmatrix} a \\ -va \end{bmatrix} + va \cdot \begin{bmatrix} -b \\ d \end{bmatrix} \quad \text{and} \quad \lambda \cdot \begin{bmatrix} 1 \\ c \end{bmatrix} = \begin{bmatrix} a \\ va \end{bmatrix} + c \cdot \begin{bmatrix} b \\ d \end{bmatrix},$$

where c is the speed of light. We deduce

$$d = a \quad \text{then} \quad va = c^2 b \quad \iff \quad b = \frac{va}{c^2}$$

and then

$$a = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad \lambda = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

A solution exists if and only if $|v| < c$.