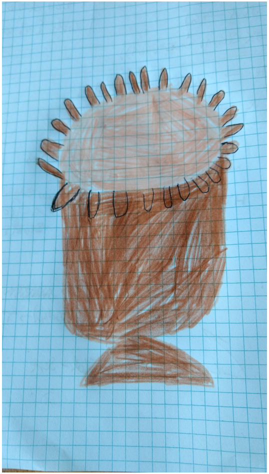
The spectral theorem



Let A be a self-adjoint operator

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

on a (finite dimensional) inner product space V.

Then V possesses an orthonormal basis of eigenvectors of A with real eigenvalues.

On Euclidean space

A real matrix A is self-adjoint with respect to the dot product

$$\vec{x} \bullet \vec{y} = \vec{x}^t \vec{y}$$

on $\mathbb{R}^n$ if and only if A is symmetric:

$$A^t = A.$$

The theorem asserts that then

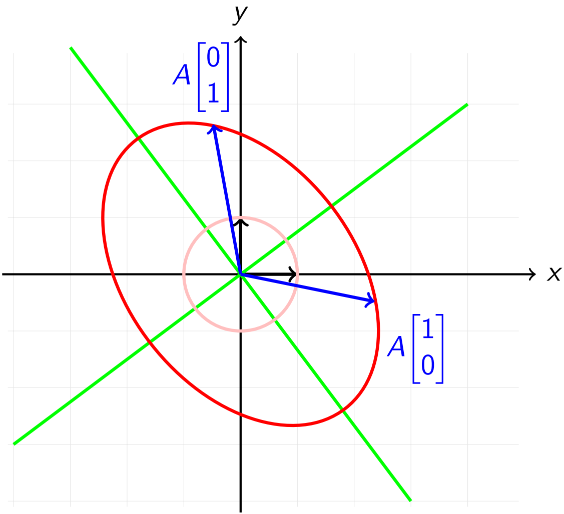
$$A = UDU^t,$$

where D is a diagonal matrix, and U is an orthogonal matrix

$$U^t = U^{-1}.$$

Example

$$\begin{bmatrix} 2.36 & -.48 \\ -.48 & 2.64 \end{bmatrix} = \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$$



How to find U and D

1. Find the roots of the characteristic polynomial $det(\lambda I - A)$. These are the eigenvalues of A.
2. For each eigenvalue λ find the corresponding eigenspace V_λ (the null-space of $\lambda I - A$).
3. For each λ, choose an orthnormal basis of V_λ, using Gram-Schmidt, if necessary.
4. Assemble them into a basis of $\mathbb{R}^n$ (i.e., choose an order).
5. Write the j^{th} basis element into the j^{th} column of U and the corresponding eigenvalue into the (j, j)-entry in D.
6. Don't confuse where to put U and where $U^{-1} = U^t$.

Multivariable second derivative test

The Hessian matrix $D^2 f$ of a smooth function

$$f \colon \mathbb{R}^n \longrightarrow \mathbb{R},$$

is symmetric at every point of $\mathbb{R}^n$,

$$(D^2 f)_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} = (D^2 f)_{j,i}.$$

To determine whether a critial point is a local maximum, a local minimum or a saddle, compute the signs of the principal curvatures (eigenvalues). The principal directions are the eigenvectors.

Proof of the spectral theorem in finite dimensions

The characteristic polymomial satisfies

$$P_{char}\left(\overline{A}^t\right) = \overline{P_{char}(A)}.$$

Hence the eigenvalues of a self-adjoint operator are real.

The orthogonal complement of an A-invariant subspace W is $\overline{A}^t$-invariant:

$$\langle w, \overline{A}^t v\rangle = \langle Aw, v\rangle = 0,$$

for $v \in W^\perp$ and $w \in W$.

Inductively, for $A = \overline{A}^t$ we obtain the decomposition

$$\mathbb{R}^n = \bigoplus_\lambda V_\lambda \qquad (\textit{orthogonally}).$$

Can you hear the length of a string?

Further reading: ‘The sound of symmetry’ by Lu and Rowlet

The Laplace operator

$$\Delta(f) = \frac{\partial^2}{\partial x^2} f$$

is self-adjoint with respect to the inner product

$$\langle f, g \rangle = \int_0^a f(x) g(x) dx$$

on the space of smooth real valued functions on the interval $[0, a]$ vanishing at the boundary

$$f(0) = 0 = f(a).$$

The spectrum of a shape

Laplace equation

$$\Delta(f) = -\lambda f$$

The spectrum of a domain (drumhead) or our string is the set of eigenvalues of the Laplace operator under the constraint $f|_{boundary} = 0$.

These are in bijection with the resonant frequencies.

Two dimensional Laplace operator:

$$\Delta = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2}$$

In one dimension

Motivation for Laplace's equation: The one dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Seperating variables

$$u(x,t) = f(x)g(t)$$

and rescaling the time variable to make $c = 1$, we get

$$\frac{f''(x)}{f(x)} = \frac{g''(t)}{g(t)} = \text{constant},$$

hence the Laplace equation. The Solutions are

$$f_k(x) = \sin\left(\frac{k\pi x}{a}\right) \qquad \lambda_k = \frac{k^2\pi^2}{a^2}.$$