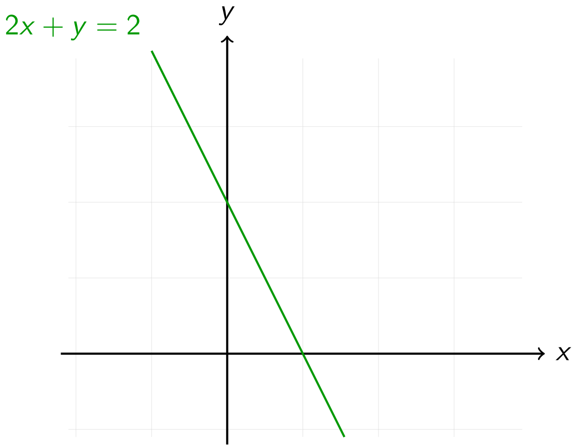
Ways to represent one linear equation in two variables



graphic representation of the solution space:

the line through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

equation form:

$2x + y = 2$

matrix form:

$\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$

dot product form:

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = 2$

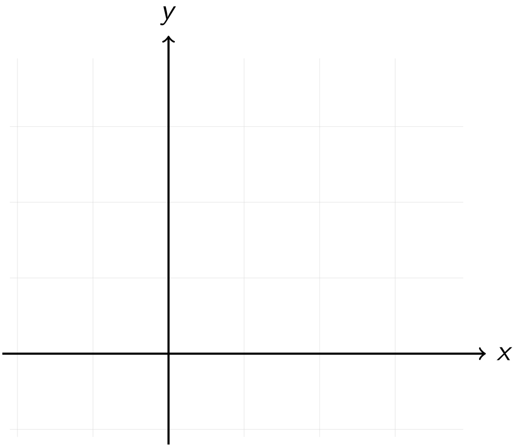
solved for x:

$x = 1 - \frac{y}{2}$

solved for y:

$y = 2 - 2x$

Another example



graphic representation of the solution space:

the line through $\begin{bmatrix} \\ \\ \end{bmatrix}$ and $\begin{bmatrix} \\ \\ \end{bmatrix}$

equation form:

$x + 3y = 2$

matrix form:

$\begin{bmatrix} & \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} \end{bmatrix}$

dot product form:

$\begin{bmatrix} \\ \\ \end{bmatrix} \bullet \begin{bmatrix} \\ \\ \end{bmatrix} =$

solved for x:

$x =$

solved for y:

$y =$

Is the solution space always a line?

The solution space of one equation in two variables x and y is almost always a line. Here are some exceptions:

$$\begin{aligned} 0x + 0y &= 0 && \text{the solution space is the whole plane} \\ 0x + 0y &= 7 && \text{no solutions} \end{aligned}$$

In fact, all exceptions are of the form

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [b]$$

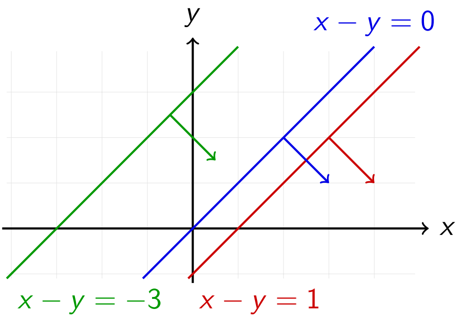
where b is a given constant. If $b = 0$, then every choice of x and y is a solution, if $b \neq 0$, then the solution space is empty.

What happens when we change the constant?

The equation

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [b]$$

for $b = -2$, for $b = 0$, and $b = 1$:



The solution space of the homogeneous equation (i.e., for $b = 0$) passes through the origin. The others are paralell to it. All three lines have normal vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

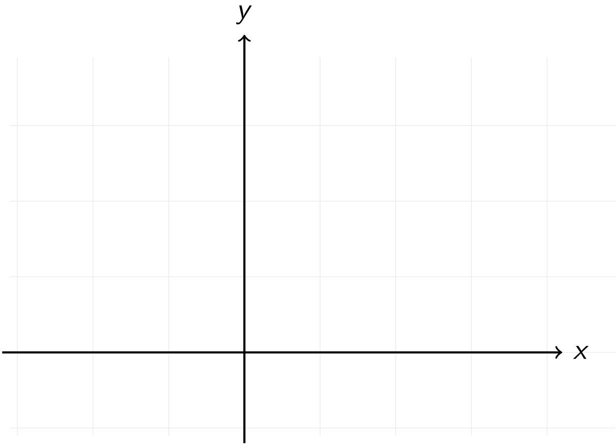
Matrix rows are to be thought of as normal vectors.

Another example

The equation

$$\begin{bmatrix} 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [b]$$

for $b = 3$, for $b = 0$, and $b = 12$.



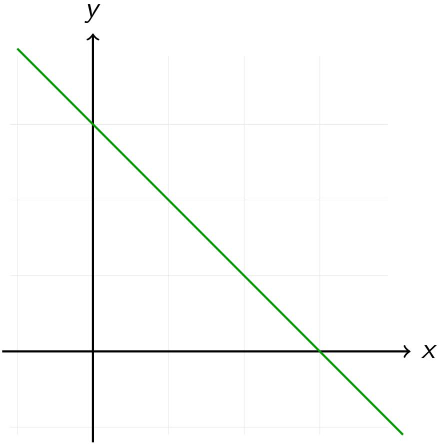
Scaling

Multiplying both sides of an equation with the same scalar does not change the solutions

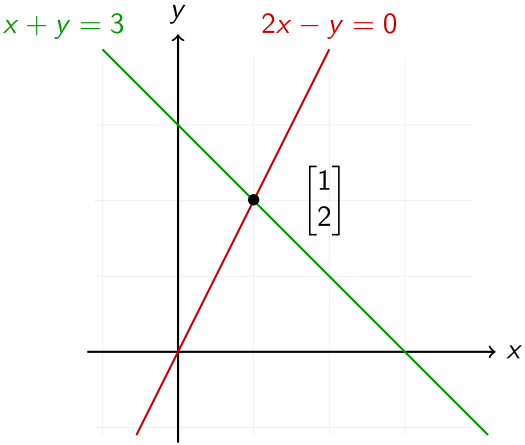
Example: The equations

$$x + y = 3 \quad \text{and} \quad 2x + 2y = 6$$

have identical solution spaces.



Two linear equations in two variables



Both equations are satisfied if and only if $x = 1$ and $y = 2$.

The solution space of the system consists of the point where the two lines intersect.

In matrix form, this system of equations is

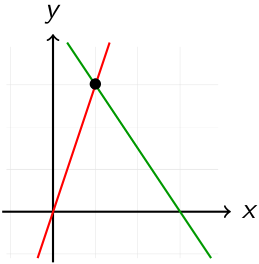
$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Row by row, this reads

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}.$$

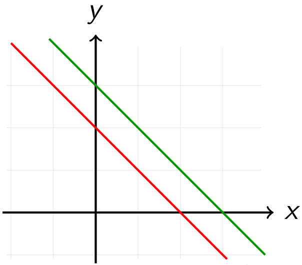
How many solutions?

There are three scenarios:



$3x + 2y = 9$

$3x - y = 0$



$x + y = 3$

$x + y = 2$

$x + y = 3$

$2x + 2y = 6$