The Lorentz Group
&
the Klein-Gordon Equation

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1 The Lorentz Group

1.1 Minkowski Space

In any inertial reference frame, the speed of light, denoted by c, is constant. This is an empirically known fact. An immediate corollary of this is: unlike the 3-dimensional Euclidean space that we perceive in our everyday lives, we actually live in a 4-dimensional manifold called **Minkowski space**[1]

$$\mathcal{M} := \left(\mathbb{R}^{1+3},\, g = \mathrm{diag}(1,-1,-1,-1)\right), \tag{1.1.1}$$

where g is the **Minkowski metric**.

The inner product between two 4-vectors $v, w \in M$ is

$$\langle v, w\rangle = v^T g w = v_0 w_0 - v_1 w_1 - v_2 w_2 - v_3 w_3,$$

or in compact indicial notation

$$\langle v, w\rangle = v^\mu g_{\mu\nu} w^\nu := v^\mu w_\mu \tag{1.1.2}$$

This also allows us to capture a notion of "length" for a 4-vector, viz

$$\|v\|^2 := v^\mu v_\mu \tag{1.1.3}$$

The above postulate of the constancy of c, together with the homogeneity of spacetime and isotropy of space, implies that this norm is independent of inertial reference

[1]This excludes the phenomenon of gravitation, in which spacetime is curved in the presence of mass and inherently non-Euclidean. In such a case, our pseudo-Euclidean Minkowski space is the tangent space to the spacetime manifold.

frame, for "physical" vectors. In a similar fashion to the orthogonal group, we define the **Lorentz group** $O(1,3)$ to be the real matrix Lie group that preserves this norm.

Suppose we make the transformation

$$x^\mu \longmapsto x'^\rho = \Lambda^\rho{}_\mu x^\mu$$

where $\Lambda \in O(1,3)$, which we represent as a 4×4 matrix. For the norm to remain invariant we require

$$x^\mu g_{\mu\nu} x^\nu \overset{!}{=} x'^\rho g_{\rho\sigma} x'^\sigma \tag{1.1.4}$$

$$= \Lambda^\rho{}_\mu x^\mu g_{\rho\sigma} \Lambda^\sigma{}_\nu x^\nu \tag{1.1.5}$$

$$\iff g_{\mu\nu} = \Lambda^\rho{}_\mu g_{\rho\sigma} \Lambda^\sigma{}_\nu \tag{1.1.6}$$

In (perhaps more familiar) matrix notation this is written

$$\Lambda^T g \Lambda = g \tag{1.1.7}$$

Compare this with our definition of orthogonal matrices, and you will see it is essentially the same, with g replaced by the identity matrix. For this reason, the Lorentz group is a type of **generalised orthogonal group** $O(p,q)$, which consist of matrices who preserve a metric with signature (p,q).[2]

1.2 Classification of the Lorentz Group

Like any Lie group, the Lorentz group can be characterised as a manifold. Let us take the determinant of both sides of (1.1.7), which yields

$$\begin{aligned} &-(\det \Lambda)^2 = -1 && \text{since } \det g = -1 \\ &\implies \det \Lambda = \pm 1. \end{aligned}$$

We see that, as with the orthogonal group, we can partition the Lorentz group into those matrices with determinant 1 and -1. In full analogy to $O(N)$, we denote the former subgroup as

$$SO(1,3) \coloneqq \{\Lambda \in O(1,3) \mid \det \Lambda = 1\} \tag{1.2.1}$$

We can, in fact, partition further. This is because, unlike for $O(N)$, we have separate spatial parity

$$(\Lambda_P : x \mapsto -x) \in O(1,3)$$

and time reversal

$$(\Lambda_T : t \mapsto -t) \in O(1,3)$$

[2]This means that the metric consists of a positive identity block of size p and a negative identity block of size q, along the diagonal. Some authors use the opposite sign convention, and this is permissible since it only introduces factors of -1 here and there.

We will demonstrate shortly that neither of them are continuously connected to the identity. So the Lorentz group exhibits another bifurcation of elements: those with $\Lambda^0_0 \geq 1$, and those with $\Lambda^0_0 \leq -1$. A brief derivation of the inequality can be found in Schwichtenberg's comprehensive book [4].

We observe that the identity has the properties

$$\det \mathrm{Id} = +1; \quad \mathrm{Id}^0{}_0 = +1 \tag{1.2.2}$$

Since we want to consider the component of $O(1,3)$ that is continuously connected to the identity, we therefore turn our attention to the following subgroup:

$$SO^+(1,3) \coloneqq \{\Lambda \in O(1,3) \,|\, \det \Lambda = +1, \Lambda^0_0 \geq 1\} \tag{1.2.3}$$

This is called the **proper orthochronous Lorentz group**, or simply the **restricted Lorentz group**. It is dubbed proper since it has positive determinant, and orthochronous since it preserves the direction of time. The complete Lorentz group can be understood via the relation

$$O(1,3)/SO^+(1,3) = \{1, \Lambda_P, \Lambda_T, \Lambda_P\Lambda_T\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{1.2.4}$$

One way to observe why the Lorentz group consists of four disconnected components is to see the explicit representations for Lorentz transformations acting on 4-vectors. Physically the Lorentz transformations consist of rotations in 3-dimensional space, as well as "boosts" along some specified axis, whereby two coordinate systems are separated via some constant velocity. We can simply write any rotation as

$$\Lambda_{ROT} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \tag{1.2.5}$$

where $R \in SO(3)$, since elements of this form indeed satisfy the condition of leaving the norm invariant.

The boosts take on a slightly different form. We simply quote them here; a more detailed consideration can be found in [4]. We have

$$B_x(\xi) = \begin{pmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.2.6}$$

$$B_y(\xi) = \begin{pmatrix} \cosh\xi & 0 & \sinh\xi & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\xi & 0 & \cosh\xi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.2.7}$$

$$B_z(\xi) = \begin{pmatrix} \cosh\xi & 0 & 0 & \sinh\xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \sinh\xi & 0 & 0 & \cosh\xi \end{pmatrix} \tag{1.2.8}$$

where $\sinh \xi$ and $\cosh \xi$ denote the **hyperbolic sine** and **hyperbolic cosine**. They satisfy

$$\cosh^2 \xi - \sinh^2 \xi = 1$$

The parameter ξ is referred to as the **rapidity**. Physically, it is related to the velocity via

$$\xi = \operatorname{artanh} \beta \tag{1.2.9}$$

where $\beta \coloneqq v/c$. Let $\gamma = \cosh \xi$. Then a boost in the x direction, in $(1+1)$ dimensions for brevity, takes on the form

$$\begin{aligned} B_x(\xi)(t,x) &= (\cosh \xi t + \sinh \xi x, \cosh \xi x + \sinh \xi t) \\ &= (\gamma(t + \beta x), \gamma(x + \beta t)) \,, \end{aligned}$$

which may be familiar to anyone who has been introduced to special relativity.[3] The γ factor is usually referred to as the Lorentz factor. It describes a dilation of spacetime between different frames.

In this representation we see that, due to the positive definiteness of the hyperbolic cosine, there is no way to attain a parity or time reversal transformation from a continuous change of the parameter ξ. Moreover, we see that $|\Lambda^0_0| \geq 1$ since $|\Lambda^0_0| = \gamma = (1-\beta^2)^{-1/2}$ and $\beta \in (-1,1)$ (we also see that $\beta = \pm 1$ leads to singular behaviour, agreeing with the fact that c is a speed limit and $-c < v < c$).

Since Lie groups permit classifications as manifolds, we can visualise this disconnectedness by observing that $SO(2)$ is topologically equivalent to a circle, whereas $SO(1,1)$ is topologically a hyperbola, which is manifestly not connected. This readily generalises to a similar discrepancy between $SO(4)$ and $SO(1,3)$, albeit in higher dimension.

The disconnectedness of $O(1,3)$ gives us the following picture, taken from [4]:

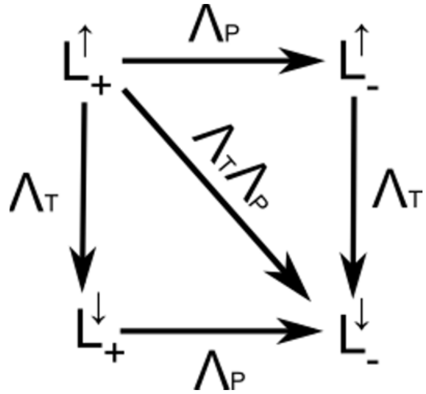

Figure 1: Here, $L^\uparrow_+$ is the notation for our $SO^+(1,3)$.

[3] A minus sign in front of β may be more familiar. This is just a matter of whether a boost is defined in the positive or negative direction.

1.3 The Lie algebra

Unless otherwise explicitly stated, we take Lie algebras in the following section as implicitly being over $\mathbb{R}$. Now that we have identified the subgroup of $O(1,3)$ which is continuously connected to the identity, $SO^+(1,3)$, we can explore its Lie algebra so that we might classify its irreducible representations. We define

$$\mathfrak{so}(1,3) \coloneqq \mathrm{Lie}\left\{SO(1,3)\right\} = \mathrm{Lie}\left\{SO^+(1,3)\right\} \tag{1.3.1}$$

Let $J_i \in \mathfrak{so}(3)$ denote the generators of the Lie algebra for $SO(3)$. By taking derivatives of the given boost matrices above, we define K_i as the boost generators. Explicitly

$$K_x = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \tag{1.3.2}$$

$$K_y = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \tag{1.3.3}$$

$$K_z = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \tag{1.3.4}$$

We thus obtain the following Lie algebra:[4]

$$[J_i, J_j] = i\varepsilon_{ijk}J_k; \quad [J_i, K_j] = i\varepsilon_{ijk}K_k; \quad [K_i, K_j] = -i\varepsilon_{ijk}J_k. \tag{1.3.5}$$

Here ε_{ijk} is the **Levi-Civita symbol**, defined by

$$\varepsilon_{ijk} \coloneqq \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3); \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3); \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.6}$$

It essentially captures handedness through its antisymmetry: $[J_i, J_j] = -[J_j, J_i]$ which has as example the cross product, e.g. $\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i}$.

We see that the K_i are not closed under commutation, so none of the generators commute with each other. Such a quality is desirable and attainable: We can define generators that partially commute if we work with the complexification of our Lie algebra, $\mathfrak{so}(1,3)_{\mathbb{C}} := \mathfrak{so}(1,3) \otimes_{\mathbb{R}} \mathbb{C}$, which allows complex scalars in our linear

[4]The factor of i here is customary, particularly in physics. It can be done without, but it has utility in quantum mechanics, where Hermitian operators are our friends.

combinations. A natural method of complexification is to define a new basis for the **complex** Lie algebra $\mathfrak{so}(1,3)_{\mathbb{C}}$ by taking specific complex-linear combinations of our old basis vectors,

$$M_i^{\pm} = \frac{J_i \pm iK_i}{2}.$$

It can be checked that $J_i, K_i \in \langle M_i^{\pm} \rangle_{(\mathbb{C},[\,\cdot\,,\,\cdot\,])}$, e.g. $K_i = \mathrm{i}\left(M_i^- - M_i^+\right)$. After computation, we ultimately obtain:

$$[M_i^{\pm}, M_j^{\pm}] = i\varepsilon_{ijk} M_k^{\pm}; \quad [M_i^+, M_j^-] = 0. \tag{1.3.7}$$

The first set of commutation relations for the M_i^+ or the M_i^- are identical to the commutation relations for the basis elements in $\mathfrak{su}(2)_{\mathbb{C}}$. Crucially, the fact that the M_i^+ and M_j^- commute for all i, j means that we can in some sense **decouple** the relevant Lie algebras as follows: $\langle M_i^{\pm} \rangle_{(\mathbb{C},[\,\cdot\,,\,\cdot\,])} = \langle M_i^{+} \rangle_{(\mathbb{C},[\,\cdot\,,\,\cdot\,])} \oplus \langle M_i^{-} \rangle_{(\mathbb{C},[\,\cdot\,,\,\cdot\,])}$. We thus see that the complexification of the Lie algebra for the Lorentz group consists of two **separate** copies of the complexified Lie algebra for $SU(2)$, i.e. we have the following picture:

$$\boxed{\mathfrak{so}(1,3) \hookrightarrow \mathfrak{so}(1,3)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}} \tag{1.3.8}$$

1.4 The Universal Cover

We will now demonstrate the relationship between the restricted Lorentz group $SO^+(1,3)$, and the **Möbius group** of transformations on the sphere. We wish to demonstrate that the Minkowski norm is preserved under the action of $SL(2,\mathbb{C})$. To each 4-vector $x^\mu \in \mathbf{M}$, we associate a Hermitian matrix

$$X = x^\mu \sigma_\mu = \begin{pmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{pmatrix} \tag{1.4.1}$$

where we defined $\sigma^\mu = (I, \sigma_1, \sigma_2, \sigma_3)$, a vector of Pauli matrices and the identity. We observe that the determinant of the above auxiliary matrix yields

$$\det X = x^\mu x_\mu \tag{1.4.2}$$

Consider the action

$$X \longmapsto \lambda X \lambda^\dagger, \quad \lambda \in SL(2,\mathbb{C}).$$

This preserves the determinant since $\det \lambda = 1$, and thus preserves the Minkowski norm, i.e. $SL(2,\mathbb{C})$ acts by Lorentz transformations [2]. Since $SL(2,\mathbb{C})$ is connected, it must cover the connected component of the Lorentz group, $SO^+(1,3)$. We can also observe, by considering X real diagonal, that for $\lambda, \lambda' \in SL(2,\mathbb{C})$,

$$\lambda X \lambda^\dagger = \lambda' X \lambda'^\dagger \implies \lambda' = \pm\lambda. \tag{1.4.3}$$

We therefore have an isomorphism between the restricted Lorentz group and the Möbius group:

$$SO^+(1,3) \cong PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/\mathbb{Z}_2 \tag{1.4.4}$$

Since $SL(2,\mathbb{C})$ is simply connected, we see that it is the **universal cover** of $SO^+(1,3)$. Since this is also a double cover, we have by definition,

$$SL(2,\mathbb{C}) \simeq Spin(1,3) \tag{1.4.5}$$

Recall from a few weeks ago that one has the isomorphism

$$\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2)_{\mathbb{C}} \tag{1.4.6}$$

This is shown in *Chapter 2* of Knapp's book [1].

Now, since the complexification of $\mathfrak{sl}(2,\mathbb{C})$ simply gives us

$$\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C}) \oplus i\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}), \tag{1.4.7}$$

we can now combine everything to give us the full painting of isomorphisms

$$\boxed{\begin{aligned} \mathfrak{so}(1,3) \hookrightarrow \mathfrak{so}(1,3)_{\mathbb{C}} &\cong \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} \\ &\cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} \hookleftarrow \mathfrak{sl}(2,\mathbb{C}). \end{aligned}} \tag{1.4.8}$$

1.5 Irreducible Representations of the Lorentz Group

From the above correspondence, we see that the representations of the Lorentz group Lie algebra $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$ are in one-to-one correspondence with representations of the universal covering group $SL(2,\mathbb{C})$, which in turn correspond to the representations of $\mathfrak{su}(2)$. But we already know the irreducible representations of $\mathfrak{su}(2)$!

Recall that we can label the $(2j+1)$-dimensional irreducible representations of $\mathfrak{su}(2)$ by the half-integer j corresponding to the eigenvalue $j(j+1)$ of the **quadratic Casimir element**. Thus, we label the representations of the Lorentz group by a **pair** (m,n) of half-integers. In the following section, we will consider the **trivial representation** $(0,0)$, also known as the **scalar representation**.

2 The Klein-Gordon Equation

2.1 Transformation of a Scalar Field

In the trivial representation, Lorentz transformations act on **Lorentz scalars** by simply doing nothing, i.e. a Lorentz scalar is an **invariant** quantity. We have already seen an example of such a scalar, viz.

$$\langle x,x\rangle = x^\mu x_\mu \tag{2.1.1}$$

An arbitrary Lorentz scalar α transforms as

$$\alpha \xmapsto{\Lambda} \alpha' = \alpha \tag{2.1.2}$$

Thus a scalar field $\varphi(x^\mu)$, a scalar that is a function of spacetime, transforms as so:

$$\varphi(x^\mu) \xmapsto{\Lambda} \varphi'\left(x'^\mu\right) = \varphi\left(x^\mu\right) \tag{2.1.3}$$

A little bit of thought will convince you that if we wish to evaluate our transformed field at the original un-transformed point, we would have

$$\varphi'(x^\mu) = \varphi\left(\Lambda^{-1}x^\mu\right) \tag{2.1.4}$$

See Peskin & Schroeder [3] for a more thorough account.

So, we seek a relativistically **invariant** Lagrangian[5] for a scalar field. This implies that we have an invariant action, as required. With such a Lagrangian we may derive equations of motion which are covariant under Lorentz transformations.

2.2 The KGE from a Lagrangian

The most general Lagrangian we can write down is

$$\mathcal{L} = A\varphi^0 + B\varphi + C\varphi^2 + D\partial_\mu\varphi + E\partial_\mu\varphi\partial^\mu\varphi + F\varphi\partial_\mu\varphi + \ldots \tag{2.2.1}$$

We want the above to be a **scalar** quantity so that it is invariant under Lorentz transformations. We thus discard any odd powers of ∂_μ since these give rise to non-scalar quantities in the Lagrangian. We also may dispense with any constant or linear terms in φ, since these make no overall contribution to the action.[6] We can also rescale the Lagrangian however we wish, since this too has no impact on the physics. See Schwichtenberg's book [4] for more discussion.

In addition, we will only concern ourselves with a Lagrangian pertaining to a **free** field. This means we discard any interactions terms, i.e. terms of higher order than φ^2.

After suitable rescaling we are left with the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2\right). \tag{2.2.2}$$

Employing the Euler-Lagrange equation

$$0 = \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\varphi}\right) \tag{2.2.3}$$

gives the following equation of motion

$$\boxed{\partial_\mu\partial^\mu\varphi + m^2\varphi = 0} \tag{2.2.4}$$

[5] Strictly speaking we are really looking for a Lagrangian **density**. However we will allow ourselves the linguistic sin of simply referring to this as the "Lagrangian".

[6] This is true for a free field, but not necessarily the case in general.

which is the **Klein-Gordon equation**. This is the archetypal free relativistic wave equation, of similar relevance to the Schrödinger equation in non-relativisitic quantum mechanics.

As it stands, we have only introduced m^2 as some arbitrary parameter. Consideration of the full group of spacetime isometries yields the **Poincaré group**, which consists of spacetime translations appended to the Lorentz group. With this treatment, we in fact see that m corresponds to mass.

In the case of a free massless field, $m = 0$, the Klein-Gordon equation reduces to

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi = 0, \tag{2.2.5}$$

which is just the wave equation for a field propagating at light-speed. This has general solutions given by plane waves

$$\exp\left[\pm i\left(\vec{k}\cdot\vec{x} - \omega t\right)\right] \tag{2.2.6}$$

subject to the dispersion relation

$$k^2 = w^2. \tag{2.2.7}$$

This is in fact equivalent to the energy-momentum relation for $m = 0$:

$$E^2 = p^2,$$

since it is an empirical fact that the energy and momentum of a photon are

$$E = \hbar\omega; \quad \vec{p} = \hbar\vec{k},$$

where I have decided to be inconsistent and include the factor of $\hbar$.

2.3 The KGE from Quantum Operators

Historically, the derivation of the Klein-Gordon equation was motivated by a more rudimentary approach. In the same way that time and space combine together into a 4-vector x^μ, energy and momentum are combined into the 4-momentum:

$$P \coloneqq (E, \vec{p}). \tag{2.3.1}$$

This gives rise to an invariant scalar quantity, viz.

$$P^\mu P_\mu = E^2 - p^2 \coloneqq m^2. \tag{2.3.2}$$

This is in fact just the famous $E = mc^2$ formula, but we have dispensed with the factor of c^2 for simplicity.

In quantum mechanics, observable quantities are associated with Hermitian operators that act on a Hilbert space of quantum states. This process is referred to as **first quantisation**. Due to **Noether's theorem**, symmetries of the Lagrangian under the action of a given generator lead to conserved quantities. In particular we have: invariance under the generator $i\partial/\partial t$ of time translation leads to conservation of energy; invariance under the generator $-i\nabla$ of spatial translation leads to conservation of linear momentum. We thus have the associated operators:

$$E \longmapsto \hat{E} = i\frac{\partial}{\partial t}, \tag{2.3.3}$$

$$p \longmapsto \hat{p} = -i\nabla. \tag{2.3.4}$$

By simply plugging them into the energy-momentum relation above, and acting them on a field $\varphi(x^\mu)$, we yield

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\varphi = 0, \tag{2.3.5}$$

which is indeed the Klein-Gordon equation. Note that if φ is given by a Fourier series of plane waves, then the above equation reduces to $(-E^2 + p^2 + m^2)\varphi = 0$, which is equivalent to the energy-momentum relation.

Similarly, in the non-relativistic case, the energy of a free field is just the kinetic energy: $E = p^2/(2m)$. Equating the respective operators and acting them on some field Ψ yields

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2m}\nabla^2\Psi, \tag{2.3.6}$$

which is the time-dependent Schrödinger equation.

2.4 What Does It Mean?

The entire reason we are interested in the irreducible representations of the Lorentz group to begin with is because irreducible representations correspond to **fundamental particles**.

The Klein-Gordon equation is in some sense the archetypal relativistic wave equation for the scalar representation. However, the scalar representation has limited scope. It neglects any treatment of the phenomenon of spin, for example. Additionally there is only one fundamental field that is described by the scalar representation – the Higgs field – and even then there are some theories wherein the Higgs field is composite, and there are other factors which obstruct the Klein-Gordon equation from governing the Higgs field. However, there are many physical systems that can be described by an **effective** scalar field, so the Klein-Gordon equation does have some utility beyond fundamental particle physics.

For the next most rudimentary representation of the Lorentz group, which encapsulates **spin**, one would consider the $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$ representations, or the direct sum of both...

References

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