Pin and Spin Groups

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References:
[1] Representations of Compact Lie Groups, Brocker and Dieck 1985, pgs. 54-62
[2] Clifford Algebras, Clifford Groups, and a Generalization of the Quaternions: the Pin and Spin Groups, Gallier 2012

For $n \geq 1$, consider the subset $S(n) \subset O(n)$, of reflections though $n-1$ dimensional subspaces of $\mathbb{R}^n$. We have the following fact.

Proposition: The reflections $S(n)$ generate $O(n)$.

Proof: If $n = 1$, $S(n) = O(n)$, so we are done. For any n, take $A \in O(n)$. Fixing an orthonormal basis $(e_i)_{1 \leq i \leq n}$ for $\mathbb{R}^n$, choose $\sigma \in S(n)$ such that $\sigma \circ A(e_1) = e_1$. Then we can view $\sigma \circ A$ as an element of $O(n-1)$, so the proposition follows by induction. $\square$

Consider the correspondence $s : \mathbb{R}^n \setminus \{0\} \to S(n)$, which sends a non-zero vector v to $s_v \in S(n)$, which is the reflection through the hyperplane orthogonal to v. Clearly $s_v = s_u$ iff and only if $u = kv$, where $k \in \mathbb{R}^*$. If follows we can write each element of $O(n)$ as a string of non-zero vectors, up to some non-zero real multiple. We want to make this precise, in the exact sense that we want to construct an exact sequence

$$\{e\} \to \mathbb{R}^* \to \Gamma_n \to O(n) \to \{e\}$$

Where Γ consists somehow of finite strings of non-zero vectors, and our map $\rho : \Gamma_n \to O(n)$ restricts to s on $\mathbb{R}^n \setminus \{0\}$. Γ_n will be the Clifford group corresponding to certain well chosen Clifford algebras.

Last Time: We defined the Clifford algebra $Cl(V)$ for a vector space V equipped with a symmetric bilinear form $\langle -, - \rangle$ as the following quotient

$$Cl(V) := T(V)/\langle v \otimes v - \langle v, v \rangle \rangle$$

We have a canonical automorphism

$$\alpha : Cl(V) \to Cl(V)$$

Defined on basis elements by $e \to -e$. We have the anti-automorphism

$$t : Cl(V) \to Cl(V)^{opp}$$

Defined on basis elements by $e \to e$. Note that V sits as a subspace of $Cl(V)$ which is proved in [1].

For Us: $V = \mathbb{R}^n$, and our symmetric bilinear form is the negative of the standard Euclidean dot product. So for $n \geq 1$ define

$$C_n := T(\mathbb{R}^n)/\langle v \otimes v + ||v||^2 \rangle$$

Note that for $v \in \mathbb{R}^n \setminus \{0\}$, we have $v^2 = -||v||^2 \in \mathbb{R}^*$. This is good, because we intend on sending these vectors to reflections in $O(n)$ via a representation ρ such that $\ker \rho = \mathbb{R}^*$, so we better have $v^2 \in \mathbb{R}^*$ for all such vectors.

Fixing an orthonormal basis $(e_i)_{1 \leq i \leq n}$, we have the following useful relations

$$e_i^2 = -1$$

$$e_i e_j = -e_j e_i$$

We now give an equivalent definition for the Clifford group to that given last week

$$\Gamma_n := \{x \in C_n^* : (\exists v_1, ..., v_k \in \mathbb{R}^n \setminus \{0\})(x = v_1 ... v_k)\}$$

This is a well defined multiplicative subgroup of C_n. Further, we can define the following representation.

$$\rho : \Gamma_n \to \mathrm{GL}(\mathbb{R}^n)$$

Where for $x \in \Gamma_n$, $v \in \mathbb{R}^n$, $\rho_x(v) = \alpha(x) v x^{-1}$.

Lemma: ρ is a well defined linear representation of Γ_n. Further ρ restricts to s on $\mathbb{R}^n \setminus \{0\}$. It follows that $\operatorname{im} \rho = O(n)$.

Proof: Considering the representation $\bar{\rho} : \Gamma_n \to \mathrm{GL}(C_n)$, defined in the same way as ρ, this is certainly well defined, since multiplication by a unit in an algebra corresponds to a linear isomorphism, and we have for all $x, y \in \Gamma_n$, $c \in C_n$.

$$\rho_{xy}(c) = \alpha(xy) c y^{-1} x^{-1} = \rho_x \rho_y(c)$$

We show that for $v \in \mathbb{R}^n \setminus \{0\}$, $\bar{\rho}_v$ restricts to s_v on $\mathbb{R}^n$. For $v \in \mathbb{R}^n \setminus \{0\}$, we can fix an orthonormal basis $(e_i)_{1 \leq i \leq n}$ for $\mathbb{R}^n$ such that $||v|| e_1 = v$, then however, we can assume $v = e_1$, since it will act identically to v via ρ. Then we have

$$\rho_{e_1}(e_1) = -e_1$$

and

$$\rho_{e_1}(e_i) = e_i$$

Where $i \neq 1$. Then however on $\mathbb{R}^n$. $\rho_v = s_v$. It follows that $\mathbb{R}^n$ is Γ_n-invariant, and further that ρ is the correseponding subrepresentation. Then the fact that $\operatorname{im}\rho = O(n)$ is clear, since we have for $A \in O(n)$, $A = s_{v_1}...s_{v_k} = \rho_{v_1....v_k}$ where $v_i \in \mathbb{R}^n \setminus\{0\}$. Every element in the image is of this form by design, so we are done. $\square$

We give the following lemma without proof, since it uses to $\mathbb{Z}/2\mathbb{Z}$ of the Clifford algebra, which was skipped.

Lemma: $\ker\rho = \mathbb{R}^*$

So we have the exact sequence we desired

$$\{e\} \to \mathbb{R}^* \to \Gamma_n \to O(n) \to \{e\}$$

Remember we defined the norm N on a Clifford algebra to be the map $x \to x\alpha t(x) = xt\alpha(x)$. In C_n, for any $v \in \mathbb{R}^n \setminus\{0\}$ we have $N(v) = ||v||^2$. The following claim then follows simply by induction.

Lemma: $N|_{\Gamma_n} : \Gamma_n \to \mathbb{R}^*$ is a well defined homomorphism of groups.

Proof: Observe that for $v_1, ..., v_k \in \mathbb{R}^n \setminus\{0\}$, we have $N(v_1....v_k) = v_1....v_k\alpha t(v_1....v_k) = v_1....v_{k-1}N(v_k)\alpha t(v_1....v_{k-1}) = N(v_1....v_{k-1})N(v_k)$, so the proof follows by induction. $\square$

Definition: Define $\text{Pin}(n) := \ker(N|_{\Gamma_n})$.

Theorem: We have the exact sequence

$$\{e\} \to \mathbb{Z}/2\mathbb{Z} \to \text{Pin}(n) \to O(n) \to \{e\}$$

Proof: Certainly ρ restricted to $\text{Pin}(n)$ maps onto $O(n)$, since given $A \in O(n)$, we can write $A = s_{e_1}....s_{e_k} = \rho(e_1...e_k)$ where each e_i has $N(e_i) = 1$. It follows by the previous lemma that $e_1....e_k \in \text{Pin}(n)$. Further $\ker\rho|_{\text{Pin}(n)} = \ker\rho \cap \text{Pin}(n) = \{-1, 1\} \cong \mathbb{Z}/2\mathbb{Z}$

Definition: Define $\text{Spin}(n) := \rho^{-1}SO(n)$. We have immediately the exact sequence

$$\{e\} \to \mathbb{Z}/2\mathbb{Z} \to \text{Spin}(n) \to O(n)$$