The Weyl, Dirac, and Majorana Spinors

Anas and Adrian

October 2019

1 Recap and Outline

From Brendan and Brae's talks, we learned the following:

- $SU(2)$ has Lie algebra $\mathfrak{su}(2)$ which is also the Lie algebra of $SO(3)$. For each $n \in \mathbb{N}$, $SU(2)$ has a unique n-dimensional irreducible representation given by an action on bivariate n^{th} order homogeneous polynomials. Since $SU(2)$ is simply connected, each of these representations corresponds to a representation of $\mathfrak{su}(2)$ which is generated by symbols σ_i with commutation relations
$$[\sigma_i, \sigma_j] = 2\mathrm{i}\epsilon_{ijk}\sigma_k.$$
In the 2-dimensional case (i.e. spin-1/2), the symbols are called the Pauli matrices,
$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- If we complexify, we can take complex linear combinations to define generators for $\mathfrak{su}(2)_{\mathbb{C}}$,
$$H = \sigma_3, \quad X = \tfrac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2), \quad Y = \tfrac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2)$$
which have the nicer commutation relations
$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$
These are nicer because we can use the eigenvectors of H as a basis for our representation space, and think of the X and Y as raising and lowering operators.

- We saw a similar story with the Lie algebra of $SO^+(1,3)$, $\mathfrak{so}(1,3)$ (which is also the Lie algebra of $Spin(1,3)$); the real Lie algebra had 6 generators: Three rotations J_i and 3 boosts K_i with relations,
$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k, \quad [J_i, K_j] = \mathrm{i}\epsilon_{ijk}K_k, \quad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk}J_k,$$
while the complexification $\mathfrak{so}(1,3)_{\mathbb{C}}$ had generators $M_i^{\pm} := \frac{1}{2}(J_i \pm \mathrm{i}K_i)$ with the M_i^+ commuting with the M_i^- and each of them satisfying the same commutation relations as the generators σ_i of $\mathfrak{su}(2)$. Thus, $\mathfrak{so}(1,3)_{\mathbb{C}} \simeq \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$ and each representation of $\mathfrak{so}(1,3)_{\mathbb{C}}$ is labelled by a pair of half-integers (m, n). Brae investigated the $(0,0)$ representation which corresponds to scalar fields, and today we're going to explore the next-simplest.

- Upshot is that we're going to derive the Dirac spinor via an alternate route to what we saw last week. In Jackson's talk, we decided to look at the unique irreducible representation of the Dirac algebra as it coincided with a special representation of $Spin(1,3)$. We called this the $(1/2,0) \oplus (0,1/2)$ bispinor.

2 The Spinors

- So, we know that $\mathfrak{so}(1,3)_{\mathbb{C}} \simeq \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$, but we actually want to remember that we're thinking of the left $\mathfrak{su}(2)_{\mathbb{C}}$ as being generated by M_i^+ and the right one being generated by M_i^-. This has non-trivial implications for our J_i and K_i operators.
- For the $(1/2,0)$ representation, we need the M_i^+ to be 2×2 matrices that satisfy the $\mathfrak{su}(2)$ commutation relations, since our representation space is 2-dimensional. Thus, the correct choice is $M_i^+ = \sigma_i$. On the other hand, we need the M_i^- to be 1×1 matrices that satisfy the necessary commutation relations of $\mathfrak{su}(2)$, so we need $M_i^- = 0$ exactly. This means that $J_i = \mathrm{i}K_i$, and this is why it's important to assign M_i^+ to the left slot and M_i^- to the right slot, respectively! Substituting this into the definition of $M_i^+ = \sigma_i$, we get that $J_i = \sigma_i$ and $K_i = -\mathrm{i}\sigma_i$.
- If we repeat this for the $(0,1/2)$ representation, we have $M_i^+ = 0 \implies J_i = -\mathrm{i}K_i$ and that again $J_i = \sigma_i$ and $K_i = \mathrm{i}\sigma_i$. Thus, the difference in these representations is how they behave under Lorentz boosts.
- Recalling that for these representations, the σ_i are actually the 2×2 Pauli matrices, we can now construct the $(1/2,0) \oplus (0,1/2)$ representation by using block-diagonal matrices, and compare with the $S^{\mu\nu}$ defined in Jackson's talk.
- An element of the $(1/2,0)$ representation is called a left Weyl spinor and vice versa for the right Weyl spinor. One way to tell them apart is to see whether K_i acts on them like $\mathrm{i}\sigma_i$ or $-\mathrm{i}\sigma_i$. The terminology left and right has to do with handedness a.k.a. chirality. Chirality is a little difficult to explain (it essentially boils down to which representation of the Poincaré group the particle is an element of), but it coincides to the intuitive concept of helicity in special cases:

  The helicity of a particle is the projection of its spin onto its momentum; if the projection is positive, it's right-handed and vice versa for left-handed particles. This only really makes sense for particles with non-zero momentum, like photons (which are massless). To deal with particles with possibly-zero momentum, we extend the idea of helicity to chirality, which is conserved when momentum changes signs.
- Another way to see that the $(1/2,0)$ and $(0,1/2)$ representations are non-isomorphic is to compare $(1/2,0)\otimes(1/2,0)$ with $(1/2,0)\otimes(0,1/2)$ by looking at the eigenvalues of the respective operators.
- We would like our Lorentz invariant theory to abide by CPT-symmetry. That is, we want the physics to be the same if we replace all charges (e.g. electric charge, or any of the conserved charges produced from Noether's theorem) with their negatives, or apply a parity transformation ($(x,y,z) \to (-x,-y,-z)$), or apply a time reversal transformation ($t \to -t$).

It turns out that parity and time reversal transformations map $J_i \to J_i$, $K_i \to -K_i$ (we can understand this by drawing e.g. J_1 as an infinitesimal vector tangent to the unit circle on the $y-z$ plane and then replacing both axes with their negatives; for the boost K_1, we would draw an infinitesimal tangent to a hyperbola on the $t-x$ plane) and thus $M_i^+ \leftrightarrow M_i^-$. This means that a left-Weyl spinor (an element of the $(1/2, 0)$ representation) becomes a right-Weyl spinor under parity and time reversal transformations. Thus, a particle cannot be described by a left-Weyl spinor without having to deal with right-Weyl spinors simultaneously. This is why we want to discuss both by constructing a Dirac spinor as

$$\psi = \begin{bmatrix} \chi_L \\ \xi_R \end{bmatrix},$$

where χ_L is some left-Weyl spinor and ξ_R is some right-Weyl spinor.

- If we substitute this expression into the Dirac equation $(\mathrm{i}\gamma^\mu \partial_\mu - m)\psi = 0$, we obtain equations that relate χ_L and ξ_R to each other. This is another way of seeing that you can't treat a particle as a left-Weyl spinor without taking into account right-Weyl spinors. That is, unless $m = 0$, in which case the aforementioned equations decouple. Physically speaking, if an elementary particle is a Weyl spinor, it must be massless.

- We haven't really talked about charge symmetry yet. The charge conjugate of a left-Weyl spinor χ_L is $\chi_L^C := \epsilon \chi_L^*$ where $(\,\cdot\,)^*$ is complex conjugation and

$$\epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

  It turns out that χ_L^C is a right-Weyl spinor, so we call it χ_R. Likewise, the charge conjugate of a right-Weyl spinor ξ_R is $\xi_R^C = -\epsilon \xi_R^*$, which we call ξ_L. Of course, $(\chi_L^C)^C = \chi_L$ and $(\xi_R^C)^C = \xi_R$. This formula is motivated by physics and can't be explained much further here. However, it is interesting to note that a particle's anti-particle partner is its charge conjugate. Charge conjugation does not follow through to the Dirac spinor in a straightforward manner. Instead,

$$\begin{bmatrix} \chi_L \\ \xi_R \end{bmatrix}^C = \begin{bmatrix} \xi_L \\ \chi_R \end{bmatrix}.$$

- Finally, it may so happen that a Dirac spinor is of the form

$$\begin{bmatrix} \chi_L \\ \chi_R \end{bmatrix}$$

  so that the second component completely depends on the first. Then, its dimension reduces to 2 and we call it a Majorana spinor. Note that it is its own antiparticle.

- In the standard model, all known elementary fermions are spin 1/2 Dirac spinors, except the neutrino. We do not yet know if the neutrino is a Dirac spinor or a Majorana spinor.