MULTIPLICATIVE ORIENTATIONS OF KO-THEORY AND OF THE SPECTRUM OF TOPOLOGICAL MODULAR FORMS

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ABSTRACT. We describe the space of E_∞ spin orientations of KO and the space of E_∞ string orientations of tmf.

CONTENTS

1. INTRODUCTION

In this paper we prove two results attributed to us by the senior author in [Hop02]. Namely, we describe the set of components of the space of E_∞ maps from $MSpin$ to KO, and the set of components of the space of E_∞ maps from $MO\langle 8\rangle$ (also called $MString$) to tmf. As a corollary we show that the $\widehat{A}$ orientation

$$MSpin \to KO$$

of Atiyah-Bott-Shapiro [ABS64] refines to a map of E_∞ spectra, and we show that the Witten genus

$$\pi_* MString \to MF_*$$

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is the value on homotopy groups of a map of E_∞ spectra

$$MString \to tmf.$$

A precise statement of our results about KO appears in §6, while precise statements about tmf appear in §12.

This paper is intended to be part of a larger project in prepartion, assembling the collaboration of the senior author and others on the homotopy theory of elliptic cohomology and topological modular forms. We expect the manuscript to include an overview of the results presented here, and so we provide a relatively brief introduction. For more information we recommend the lectures [Hop95, Hop02], which describe the results of the larger project in a broader context.

In the course of giving a physical count of the elliptic genus of Ochanine and Landweber-Ravenel-Stong, Witten [Wit87] introduced a genus w of Spin manifolds, with KO characteristic series

$$\sigma(L,q) = (L^{1/2} - L^{-1/2}) \prod_{n\geq 1} \frac{(1-q^nL)(1-q^nL^{-1})}{(1-q^n)^2}.$$

He gave a physical argument that, if M is a *Spin* manifold and $c_2(M) = 0$, then $w(M)$ is the q-expandion of a modular form for $SL_2\mathbb{Z}$.

Note that if τ is a number in the complex upper half-plane, and $z \in \mathbb{Z}$, then setting $L = e^z$ and $q = e^{2\pi\tau}$ makes σ a holomorphic function of z, vanishing to first order at each of the points of the lattice

$$2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z}.$$

Thus σ is a form of the Weiestrass sigma function. In the [HBJ92], Hirzebruch, Berger, and Jung [HBJ92] recognized that this feature of the Witten genus gave it a special place among elliptic genera.

In [AHS01], the authors introduced the notion of an *elliptic spetrum*: this is triple (E, C, t) consisting of a complex-orientable ring spectrum E, equipped with an isomorphism

$$t : \operatorname{spf} E^0\mathbb{C}P^\infty \cong \hat{C}$$

between its formal group and the formal group of an elliptic curve C. They used Abel's Theorem–equivalently, the Theorem of the Cube–to show that that every elliptic spectrum receives a canonical map of ring spectra

$$MU\langle 6\rangle \xrightarrow{\sigma(E,C,t)} E,$$

naturally in the elliptic spectrum. For the elliptic spectrum associated to the Tate elliptic curve, the orientation is the Witten genus, in the sense that the diagram

$$\begin{tikzcd} MU\langle 6\rangle \arrow[r] \arrow[d, "\sigma(K_{Tate})"'] & MO\langle 8\rangle \arrow[d, "w"] \\ K[\![q]\!] \arrow[r] & KO[\![q]\!] \end{tikzcd}$$

commutes.

Inspired by this result, the senior author in collaboration with Goerss and Miller showed that the notion of of elliptic spectrum can be rigiditifed and enriched, into a sheaf of E_∞ spectra $\mathcal{O}_{top}$ on the moduli stack M_{Ell} of elliptic curves, equipped with an isomorphism

$$\operatorname{spf} \mathcal{O}^0_{top}\mathbb{C}P^\infty \cong \hat{\mathcal{C}}$$

(here $\mathcal{C}$ denotes the tautological elliptic curve over M_{Ell}). They defined the spectrum of *Topological Modular Forms* to be

$$tmf = \Gamma_{ho}(M_{\mathrm{Ell}}, \mathcal{O}_{top}),$$

and the result of [AHS01] clearly suggested that there should be a map of ring spectra

$$MU\langle 6\rangle \to tmf$$

or better

$$MO\langle 8\rangle \to tmf,$$

such that for any elliptic spectrum (E, C, t), the diagram

$$\begin{array}{ccc} MU\langle 6\rangle & \longrightarrow & MO\langle 8\rangle \\ \downarrow & & \downarrow \\ E & \longleftarrow & tmf \end{array}$$

should commute, and such that

$$MO\langle 8\rangle \to tmf \to KO[\![q]\!]$$

is the map associated to the Witten genus. This is the result we prove here.

The argument proceeds as follows. In §2 we recall and elaborate on the obstruction theory of May, Quinn, and Ray [MQR77]. If R is a ring spectrum, then its *space of units* is the pull-back GL_1R in the diagram

$$\begin{array}{ccc} GL_1R & \longrightarrow & R \\ \downarrow & & \downarrow \\ (\pi_0 R)^\times & \longrightarrow & \pi_0 R. \end{array}$$

It is so named because, if X is a space, then

$$[X, GL_1R] = (R^0(X))^\times.$$

If R is an A_∞ spectrum, then GL_1R has a classifying space BGL_1R, and if R is an E_∞ spectrum, then there is a spectrum gl_1R such that

$$GL_1R \approx \Omega^\infty gl_1 R.$$

Indeed, the functor gl_1R is the right adjoint up to homotopy of

$$\Sigma^\infty_+\Omega^\infty : (-1)\text{-connected spectra} \to E_\infty\text{-spectra}.$$

If S denotes the sphere spectrum, then BGL_1S is the classifying space for stable spherical fibrations.

Let

$$F : B \to BGL_1S$$

be an infinite loop space over BGL_1S: say

$$F = \Omega^\infty \left(b \xrightarrow{f} \sigma gl_1 S \right).$$

It is convenient to desuspend this once and consider it as a map

$$j : g = \Sigma^{-1} b \to gl_1 S.$$

Then the Thom spectrum M of F is an E_∞ spectrum, and, if R is an E_∞ spectrum, then the obstruction to giving an E_∞ orientation

$$M \to R$$

is the horizontal composition in

$$\begin{tikzcd} g \arrow[r, "j"] & gl_1 S \arrow[r, "\iota"] \arrow[d] & gl_1 R, \\ & Cj \arrow[ur, dashed] & \end{tikzcd} \tag{1.1}$$

and the space of E_∞ orientations is just the space of indicated factorizations. For string orientations of tmf, we can take $R = tmf$ and $g = \Sigma^{-1} bo\langle 8\rangle = \Sigma^7 bo$.

We eventually replace both the source and target in the mapping problem above. For example, let us suppose that R is E_n-local. In §4.4 we show that the fiber of

$$\pi_q \operatorname{fib}(gl_1 R \to L_n gl_1 R)$$

is torsion, and vanishes for $q > n$. This allows us to replace $gl_1 tmf_p^\wedge$ with $L_{K(1)\vee K(2)} gl_1 tmf_p^\wedge$, and so avail ourselves of the homotopy pull-back square

$$\begin{array}{ccc} gl_1 tmf_p^\wedge & \longrightarrow & L_{K(2)} gl_1 tmf_p^\wedge \\ \big\downarrow & & \big\downarrow \\ L_{K(1)} gl_1 tmf_p^\wedge & \longrightarrow & L_{K(1)} L_{K(2)} gl_1 tmf_p^\wedge. \end{array} \tag{1.2}$$

Next, Bousfield ($n = 1$) and Kuhn (general n) have shown that $L_{K(n)} X$ is a functor of $\Omega^\infty X$, and this implies that

$$L_{K(n)} gl_1 tmf_p^\wedge \approx L_{K(n)} tmf_p^\wedge,$$

and so the square (1.2) becomes

$$\begin{array}{ccc} gl_1 tmf_p^\wedge & \longrightarrow & L_{K(2)} gl_1 tmf_p^\wedge \\ \big\downarrow & & \big\downarrow \\ L_{K(1)} gl_1 tmf_p^\wedge & \longrightarrow & L_{K(1)} L_{K(2)} gl_1 tmf_p^\wedge. \end{array}$$

Similarly, the theorem of Bousfield and Kuhn allows us to use the Adams Conjecture, to replace the source Cj in the mapping problem above with $bo\langle 8\rangle$. Since $bo\langle 8\rangle$ is $K(2)$-acyclic and $L_{K(1)} bo\langle 8\rangle \approx KO_p$, we end up with a sequence

$$[bo\langle 8\rangle, gl_1 tmf_p^\wedge] \to [KO_p, L_{K(1)} tmf_p^\wedge] \xrightarrow{A} [KO_p, L_{K(1)} L_{K(2)} tmf_p^\wedge]. \tag{1.3}$$

Work of Adams, Harris, and Switzer [AHS71] and results about $L_{K(1)} tmf_p^\wedge$ imply that $[bo\langle 8\rangle, L_{K(1)} tmf_p^\wedge]$ is the set of measures on $\mathrm{cts}(\mathbb{Z}_p^\times / \pm 1, \mathbb{Z}_p)$ taking values in p-adic modular forms. These in turn can be identified with the set of sequences of p-adic modular forms satisfying a generalization of the Kummer congruences.

Using the "logarithm" of [Rez06], we identify the kernel of the map A with those sequences of p-adic modular forms which satisfy the Kummer congruences and are in the kernel of the Atkin operator.

Finally, the maps $bo\langle 8\rangle \to L_{K(1)} tmf_p^\wedge$ which solve the mapping problem (1.1) are sequences g_k of p-adic modular forms which satisfy the generalized Kummer congruences, are in the kernal of the Atkin operator, and satisfy

$$g_k \equiv G_k \mod \mathbb{Z}[\![q]\!],$$

where G_k is tne Eisenstein series, normalized so that

$$G_k(q) = -\frac{B_k}{2k} + o(q).$$

Again, a precise statement of our results concerning tmf is given in §12.

2. Units of ring spectra and the space of orientations

2.1. The spectrum of units and E_∞ orientations. In this section we recall and elaborate on the obstruction theory for E_∞ orientations of May, Quinn, and Ray [MQR77]. We shall be brief, as a more detailed account is given in another paper [ABMR].

Definition 2.1. If R is a ring spectrum, then the *space of units* of R is the space $GL_1 R$ which is the homotopy pull-back in the diagram

$$\begin{array}{ccc} GL_1 R & \longrightarrow & \Omega^\infty R \\ \big\downarrow & & \big\downarrow \\ (\pi_0 R)^\times & \longrightarrow & \pi_0 R. \end{array}$$

If X is a space, then

$$[X, GL_1 R] = R^0(X_+)^\times,$$

and if X is a pointed space, then

$$[X, GL_1 R]_+ = (1 + \tilde{R}^0(X))^\times \subseteq R^0(X_+)^\times.$$

If R is an E_∞ spectrum, then there is a spectrum $gl_1 R$ such that $GL_1 R \approx \Omega^\infty gl_1 R$, and gl_1 is the right adjoint up to homotopy of the functor

$$\Sigma^\infty_+ \Omega^\infty \overset{\text{def}}{=} \Sigma^\infty_+ \Omega^\infty : (-1)\text{-connected spectra} \to E_\infty\text{-spectra}.$$

Indeed, one can choose topological model categories of (-1)-connected spectra and E_∞-spectra so that one has the following.

Theorem 2.2. *The functors $\Sigma^\infty_+ \Omega^\infty$ and gl_1 model adjunctions*

$$\Sigma^\infty_+ \Omega^\infty : \operatorname{Ho}((-1)\text{-}\textit{connected spectra}) \rightleftarrows \operatorname{Ho} E_\infty\text{-spectra} : gl_1.$$

□

Example 2.3. If S is the sphere spectrum, then $GL_1 S$ is the components of QS^0 of degree ± 1, and $BGL_1 S$ is the classifying space for stable spherical fibrations.

Suppose that b is a spectrum over $bgl_1 S = \Sigma gl_1 S$, participating in a triangle

$$\Sigma^{-1} b \xrightarrow{\Sigma^{-1} j} gl_1 S \to C(j) \to b \xrightarrow{f} bgl_1 S.$$

For convenience let $g = \Sigma^{-1} b$. Note that if $B = \Omega^\infty B$, then after looping down we have a map

$$B \to BGL_1 S \tag{2.4}$$

and so a stable spherical fibration over B.

Definition 2.5. The Thom spectrum of $f : b \to bgl_1 S$ is the homotopy pushout $M = Mf$ in the diagram of E_∞ spectra

$$\begin{tikzcd} \Sigma^\infty_+ \Omega^\infty (gl_1 S) \arrow[r] \arrow[d] & S \arrow[d] \\ \Sigma^\infty_+ \Omega^\infty (Cf) \arrow[r] & M. \end{tikzcd} \tag{2.6}$$

The spectrum underlying Mf is the usual Thom spectrum of the spherical fibration classified by (2.4).

Now suppose that R is an E_∞ spectrum with unit $\iota : S \to R$. The description (2.6) of M, together with the adjunction between $\Sigma^\infty_+ \Omega^\infty$ and gl_1, shows that the space $E_\infty(M, R)$ is naturally weakly equivalent to the homotopy pull-back in the diagram

$$\begin{tikzcd} E_\infty(M, R) \arrow[r] \arrow[d] & \text{spectra}(Cf, gl_1 R) \arrow[d] \\ \{i\} \arrow[r] & \text{spectra}(gl_1 S, gl_1 R). \end{tikzcd} \tag{2.7}$$

Let $bgl_1 R = \Sigma gl_1 R$, and let p be the fiber in

$$p \to bgl_1 S \to bgl_1 R.$$

It is useful to consider the diagram

$$\begin{tikzcd} gl_1 S \arrow[r] \arrow[d, equal] & Cf \arrow[r] \arrow[d, dashed] & b \arrow[r, "f"] \arrow[d, dashed] & bgl_1 S \arrow[d, equal] \\ gl_1 S \arrow[r, "i"] & gl_1 R \arrow[r] & p \arrow[r] & bgl_1 S, \end{tikzcd}$$

which shows that we also have a homotopy pull-back diagram

$$\begin{tikzcd}
E_\infty(M,R) \arrow[r] \arrow[d] & \mathrm{spectra}(b,p) \arrow[d] \\
\{f\} \arrow[r] & \mathrm{spectra}(b, bgl_1 S).
\end{tikzcd}$$

In other words, the space of E_∞ orientations is the space of lifts in the fibration

$$\begin{tikzcd}
& gl_1 R \arrow[r, equal] \arrow[d] & gl_1 R \arrow[d] \\
& p \arrow[r] \arrow[d] & * \arrow[d] \\
b \arrow[r, "f"'] \arrow[ur, dashed] & bgl_1 S \arrow[r] & bgl_1 R.
\end{tikzcd}$$

2.2. **The space of units and orientations.** In this form, there is an analogous unstable result. If R is merely an A_∞ spectrum, then one can still form the delooping BGL_1R, and indeed there is a fibration

$$GL_1R \to EGL_1R \to BGL_1R.$$

Suppose that $F : B \to BGL_1S$ is a map of spaces, and let P be the pull-back in the diagram

$$\begin{tikzcd}
& P \arrow[r] \arrow[d] & EGL_1 R \arrow[d] \\
B \arrow[r, "F"'] \arrow[ur, dashed] & BGL_1 S \arrow[r, "BGL_1\iota"] & BGL_1 R.
\end{tikzcd} \tag{2.8}$$

Let $M = MF$ be the Thom spectrum of the spherical fibration classified by F. In general it is not a ring spectrum, but it is a $\Sigma^\infty_+ X$-comodule via the relative diagonal

$$M \to \Sigma^\infty_+ X \wedge M.$$

Definition 2.9. An orientation of M in R-theory is a map of spectra

$$u : M \to R$$

such that the composition

$$M \wedge R \to \Sigma^\infty_+ X \wedge M \wedge R \xrightarrow{1\wedge u \wedge 1} \Sigma^\infty_+ X \wedge R \wedge R \to \Sigma^\infty_+ X \wedge R$$

is a weak equivalence. The *space* of orientations is the subspace of

$$u \in \mathrm{spectra}(M,R) \cong (\Sigma^\infty_+ X\text{-comodules}, R\text{-modules})(M \wedge R, \Sigma^\infty_+ X \wedge R)$$

satisfying this condition.

About this situation there is the following.

Proposition 2.10. (1) *The space of orientations $M \to R$ is naturally weakly equivalent to the space of sections in* (2.8)*.*
(2) *If $F = \Omega^\infty f$ then MF is the spectrum underlying the E_∞ spectrum Mf.*
(3) *If $u : Mf \to R$ is an E_∞ map, associated to a map*

$$t : b \to p,$$

then the underlying orientation $MF \to R$ is the one associated to

$$\Omega^\infty t : B \to P.$$

□

2.3. **The space of orientations as a torsor.** If $E_\infty(M, R)$ is non-empty, then $\pi_0 E_\infty(M, R)$ is a torsor for

$$\pi_0 E_\infty(\Sigma^\infty_+ B, R) \cong \pi_0 \operatorname{spectra}(b, gl_1 R),$$

by the E_∞ map

$$M \to \Sigma^\infty_+ B \wedge M.$$

This has the following description in the theory of units. Suppose given a diagram of spectra

$$\begin{tikzcd}
g \arrow[r, "j"] & U \arrow[r, "f"] \arrow[d, "i"'] & V \arrow[r] \arrow[dl, dashed, "u"] & b \\
& X
\end{tikzcd} \tag{2.11}$$

in which the row is a cofiber sequence, and let A be the homotopy pull-back in the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \operatorname{spectra}(V, X) \arrow[d] \\
\{i\} \arrow[r] & \operatorname{spectra}(U, X).
\end{tikzcd}$$

By construction, $[b, X]$ acts on $\pi_0 A$, and we have the following.

Lemma 2.12. *If there is a map u making the diagram* (2.11) *commute (in the homotopy category), then $\pi_0 A$ is a torsor for $[b, X]$, and a choice of map u determines a weak equivalence*

$$\operatorname{spectra}(b, X) \approx A,$$

which induces a trivialization of torsors upon applying π_0.

Proof. Let $k = ij : g \to X$. A map $u : V \to X$ as in (2.11) determines a wedge decomposition

$$Ck \approx X \vee Cf$$

making the diagram

$$\begin{tikzcd}
g \arrow[r, equal] \arrow[d, "j"'] & g \arrow[r] \arrow[d, "k"] & * \arrow[d] \\
U \arrow[r, "i"] \arrow[d] & X \arrow[r] \arrow[d] & Ci \arrow[d, equal] \\
V \arrow[r] \arrow[ur, "u"] \arrow[d] & X \vee b \arrow[r] \arrow[d] & Ci \arrow[d] \\
b \arrow[r, equal] & b \arrow[r] & *,
\end{tikzcd}$$

in which the rows and columns are cofibrations, commute. Applying spectra$(- , X)$ gives a commutative diagram

$$\begin{tikzcd}
\operatorname{spectra}(b, X) \arrow[rr, "\approx"] \arrow[dr] \arrow[dd] & & A \arrow[dr] \arrow[dd] & \\
& \operatorname{spectra}(Ck, X) \arrow[rr] \arrow[dd] & & \operatorname{spectra}(V, X) \arrow[dd] \\
\{\mathrm{id}\} \arrow[rr] \arrow[dr] & & \{i\} \arrow[dr] & \\
& \operatorname{spectra}(X, X) \arrow[rr] & & \operatorname{spectra}(U, X),
\end{tikzcd}$$

in which the indicated squares are homotopy pull-backs, and the left oblique square is obtained from the right one by base change. In particular, if $\pi_0 A$ is non-empty, then it is a torsor for $\pi_0 \operatorname{spectra}(b, X)$. $\square$

3. Rational orientations and characteristic classes

In this section we express some classical results about orientations, particularly Hirzebruch's theory of multiplicative sequences and Miller's universal Bernoulli numbers, in terms of the obstruction theory in §2.

3.1. **Rational units.** If R is a ring spectrum and X is a connected pointed finite CW complex, then there is a natural transformation

$$\begin{aligned} \log : (1 + \tilde{R}^0(X))^\times &\to \tilde{R}^0(X; \mathbb{Q}) \\ 1 + z &\mapsto \log(1+z); \end{aligned} \tag{3.1}$$

note that the induced map on homotopy groups

$$\pi_k GL_1 R \cong \widetilde{GL_1 R}^0(S^k) \xrightarrow{\log} \pi_k R \otimes \mathbb{Q}$$

is just the natural map for $k \geq 1$, and so is an isomorphism if R is rational.

This natural transformation is represented by an H-map

$$\log : GL_1 R\langle 1\rangle \to (R \otimes \mathbb{Q})\langle 1\rangle \tag{3.2}$$

which is a weak equivalence if R is rational.

If R is an E_∞ spectrum, then the map (3.2) refines to a map of spectra, and so we have the following.

Lemma 3.3. *If R is an E_∞ ring spectrum, then the logarithm* (3.1) *arises from a map of spectra*

$$gl_1(R)\langle 1\rangle \to (R \otimes \mathbb{Q})\langle 1\rangle \tag{3.4}$$

which induces the natural inclusion on homotopy groups. In particular if R is rational then this map is a weak equivalence. In geneneral, the map induces weak equivalences

$$gl_1(R \otimes \mathbb{Q})\langle 1\rangle \approx (gl_1 R)\langle 1\rangle \otimes \mathbb{Q} \approx (R \otimes \mathbb{Q})\langle 1\rangle.$$

$\square$

Since $\pi_0 gl_1 S = \{\pm 1\}$ and $S \otimes \mathbb{Q} \approx H\mathbb{Q}$, we have the following.

Corollary 3.5. *$gl_1 S \otimes \mathbb{Q}$ is contractible.* $\square$

3.2. **The Miller invariant.** Since $gl_1(S) \otimes \mathbb{Q} \approx *$, the natural map

$$gl_1(S) \otimes \mathbb{Q}/\mathbb{Z} \to bgl_1 S$$

is an equivalence.

Definition 3.6. Suppose that

$$f : b \to bgl_1 S$$

is a map of spectra, and $i : gl_1 S \to X$ is a spectrum under $gl_1 S$. The *(stable) Miller invariant* associated to f and i is the composition

$$m(f,i) : b \xrightarrow{f} bgl_1 S \underset{\approx}{\leftarrow} gl_1 S \otimes \mathbb{Q}/\mathbb{Z} \to X \otimes \mathbb{Q}/\mathbb{Z}.$$

When the maps f and i are understood from the context, we shall write $m(b,X)$ for $m(f,i)$, etc. If R is an E_∞ spectrum, then we shall write $m(b,R)$ for

$$m(b, gl_1 R) : b \to gl_1 R \otimes \mathbb{Q}/\mathbb{Z}.$$

If $F : B \to BGL_1 S$ is a map of spaces, and R is a ring spectrum, then the *(unstable) Miller invariant* associated to F and R is the composition

$$M(F,R) : B \xrightarrow{F} BGL_1 S \underset{\approx}{\leftarrow} GL_1 S \otimes \mathbb{Q}/\mathbb{Z} \to GL_1 R \otimes \mathbb{Q}/\mathbb{Z}.$$

By construction we have

$$M(\Omega^\infty b, R) = \Omega^\infty m(b, R) \in [\Omega^\infty b, GL_1 R \otimes \mathbb{Q}/\mathbb{Z}], \tag{3.7}$$

when R is an E_∞ spectrum.

The terminology recognizes the fact that, when $BU \to BGL_1 S$ is the standard map, the effect on homotopy groups of $M(BU, R)$ is given by the "universal Bernoulli numbers" introduced by Miller [Mil82]. In order to explain this, we recall Hirzebruch's theory of multiplicative sequences.

3.3. **Hirzebruch's characteristic series.** Suppose that R is a ring spectrum and V is a virtual vector bundle (or spherical fibration) over B. For simplicity we suppose that V has rank 0. A Thom class is an element $U \in R^0(B^V)$ which freely generates $R^*(B^V)$ as an $R^*(B_+)$-module. Two Thom classes $U_0, U_1 \in R^0(B^V)$ determine a *difference class*

$$\frac{U_0}{U_1} = \delta(U_0, U_1) \in R^0(B_+)^\times \tag{3.8}$$

by the formula

$$U_0 = \delta(U_0, U_1) U_1.$$

Put another way, if V is orientable in R-theory, then the set of orientations is a torsor for $R^0(B_+)^\times$.

If R is an A_∞ spectrum, this class has a simple description in the theory of units. By Proposition 2.10, the two Thom classes correspond to lifts in the digram

$$\begin{tikzcd}
 & & EGL_1 R \arrow[d] \\
B \arrow[r, "V"'] \arrow[urr, shift left, "U_0"] \arrow[urr, shift right, "U_1"'] & BGL_1 S \arrow[r] & BGL_1 R,
\end{tikzcd}$$

and so their difference is a map

$$\tilde{\delta}(U_0, U_1) \in [B, GL_1 R] \cong R^0(B_+)^\times.$$

Proposition 3.9. *The two constructions of difference class given above coincide; that is,*

$$\delta(U_0, U_1) = \tilde{\delta}(U_0, U_1) \in R^0(B_+)^\times.$$

□

If V is an oriented vector bundle and R is a rational homotopy-commutative ring spectrum, then we always have the orientation

$$\alpha : B^V \to MSO \to H\mathbb{Q} \approx S \otimes \mathbb{Q} \to R.$$

It follows that an R-orientation

$$\beta : B^V \to R$$

may be studied using the difference class

$$\delta(\alpha, \beta) \in R^0(B_+)^\times \cong H^0(B_+; R_*)^\times.$$

Hirzebruch's theory of multiplicative sequences describes the difference class in the case that $B = BSO$ (or BU, etc.). For simplicity we consider the case of BSO.

Let L be the tautological line bundle over $\mathbb{C}P^\infty$, let $U_\alpha L \in H^2((\mathbb{C}P^\infty)^L)$ be its standard Thom class in ordinary cohomology; and let $x = e_H L = \zeta^* U_\alpha L \in H^2(\mathbb{C}P^\infty_+)$ be its euler class.

Suppose $\beta : MSO \to R$ is another map of ring spectra, and let

$$U_\beta L \in R^2((\mathbb{C}P^\infty)^L)$$

be the Thom class of the tautological line bundle.

Definition 3.10. The *Hirzebruch series* of the orientation β is the difference class

$$K_\beta(x) \overset{\text{def}}{=} \delta(U_\alpha L, U_\beta L) = 1 + o(x) \in R^0(\mathbb{C}P^\infty_+)^\times \cong H^0(\mathbb{C}P^\infty_+; R_*)^\times.$$

If F denotes the formal group law over R_* classified by $MU_* \to MSO_* \to R_*$, then

$$K_\beta(x) = \frac{x}{\exp_F(x)}.$$

One then has the following.

Proposition 3.11. *The difference class*

$$\delta(\alpha, \beta) \in R^0(BSO_+)^\times$$

is the characteristic class of virtual oriented bundles whose value on a sum $L_1 \oplus \cdots \oplus L_r$ of complex line bundles is

$$\prod_i K_\beta(c_1 L_i).$$

□

3.4. **Homotopy groups.** We continue to suppose that R is a rational homotopy-commutative ring spectrum, and that we are given a homotopy multiplicative orientation

$$\beta : MSO \to R.$$

The difference class $\delta(\alpha, \beta)$ determines a pointed map

$$BU \to BSO \xrightarrow{\delta(\alpha,\beta)} GL_1 R\langle 1\rangle.$$

Let c_β be the composition

$$c_\beta : BU \to GL_1 R\langle 1\rangle \xrightarrow{\log} \Omega^\infty R\langle 1\rangle,$$

where the logarithm is the map (3.2) representing the natural transformation

$$(1 + \tilde{R}^0(X))^\times \xrightarrow{1+z\mapsto \log(1+z)} (\tilde{R}^0(X))$$

and inducing the identity on homotopy groups in positive degrees. Define classes $t_k \in \pi_{2k} R$ by the formula

$$K_\beta(x) = \exp\left(\sum_{k\geq 1} \frac{t_k}{k!} x^k\right).$$

In this section we prove the following result.

Proposition 3.12. *If v denotes the periodicity element $v = 1 - L \in K^0(S^2) \cong \pi_2 BU$, then*

$$(c_\beta)_*(v^k) = (-1)^k t_k$$

Proof. The composition

$$S^{2k} \to (\mathbb{C}P^\infty)^{\wedge k} \xrightarrow{\prod(1-L_i)} BU,$$

where the first map is the inclusion of the bottom cell, represents v^k. For $I \subseteq \{1, \ldots, k\}$ write

$$L^I = \prod_{i\in I} L_i$$

and

$$x_I = \sum_{i\in I} x_i.$$

Then

$$\prod(1 - L_i) = \sum_{I\subseteq\{1,\ldots,k\}} (-1)^{|I|} L^I,$$

and so the composition

$$(\mathbb{C}P^\infty)^{\wedge k} \xrightarrow{\prod(1-L_i)} BU \xrightarrow{\delta} GL_1 R\langle 1\rangle$$

represents the element

$$f = \prod_{I\subseteq\{1,\ldots,k\}} K_\beta(x_I)^{(-1)^{|I|}} \in (1 + \tilde{R}^0((\mathbb{C}P^\infty)^{\wedge k}) \subseteq R^0((\mathbb{C}P^\infty)^k_+)^\times. \tag{3.13}$$

If we write

$$f = 1 + ax_1 \cdot \dots \cdot x_k + o(k+1),$$

then $(c_\beta)_*(v^k) = a$.

It is easy to check that the coefficient of $x_1 \cdot \dots \cdot x_k$ in (3.13) is the coefficient of $x_1 \cdot \dots \cdot x_k$ in

$$(-1)^k \frac{t_k}{k!}(x_1 + \dots + x_k)^k,$$

which is $(-1)^k t_k$. $\square$

3.5. **The Miller invariant and universal Bernoulli numbers.** Now suppose that R is a A_∞ spectrum which is also homotopy commutative. Suppose that

$$F : B \to BSO$$

is a map, and let M be the associated Thom spectrum. By Proposition 2.10, an orientation

$$\beta : M \to R$$

corresponds to a lift in the diagram

$$\begin{tikzcd} & EGL_1R \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & BGL_1R. \end{tikzcd}$$

Once again we have the standard orientation

$$\alpha : M \to BSO \to H\mathbb{Q} \approx S \otimes \mathbb{Q} \to R \otimes \mathbb{Q},$$

and so we can consider the difference class

$$\delta = \delta(\alpha, \beta) : B \to GL_1R \otimes \mathbb{Q}.$$

If $\beta' : M \to R$ is another orientation, then

$$\delta(\alpha,\beta)\delta(\alpha,\beta')^{-1} = \delta(\beta',\beta)^{-1}$$

factors through $GL_1R \to GL_1R \otimes \mathbb{Q}$, and so the composition

$$B \xrightarrow{\delta} GL_1R \otimes \mathbb{Q} \to GL_1R \otimes \mathbb{Q}/\mathbb{Z} \tag{3.14}$$

is independent of the choice of orientation, and it is not difficult to check that

$$\delta_u = M(B, R).$$

With this observation, Proposition 3.12 gives the following.

Proposition 3.15. *If*

$$\beta : MU \to R$$

is a multiplicative map with Hirzebruch series

$$K_\beta(x) = \frac{x}{\exp_F(x)} = \exp\left(\sum_{k\ge1} t_k \frac{x^k}{k!}\right),$$

then

$$M(BU, R)_* v^k = (-1)^k t_k \mod \mathbb{Z} \in \pi_{2k}R \otimes \mathbb{Q}/\mathbb{Z}.$$

$\square$

Corollary 3.16. *If β' is another multiplicative orientation with characteristic series*

$$K_{\beta'}(x) = \exp\left(\sum_{k\geq 1} t'_k \frac{x^k}{k!}\right),$$

then

$$t_k \equiv t'_k$$

in $\pi_{2k}R \otimes \mathbb{Q}/\mathbb{Z}$.

Remark 3.17. Miller [Mil82] proves a similar result for the sequence b_k defined by

$$\frac{x}{\exp_F(x)} = \sum b_k x^k.$$

In our applications, we need the following generalization of Proposition 3.15. We still suppose that R is homotopy-commutative A_∞ ring spectrum, and we suppose given a multiplicative orientation

$$\beta : MO\langle 2n\rangle \to R.$$

After rationalizing we may consider the difference class

$$\delta(\alpha, \beta) : BO\langle 2n\rangle \to GL_1(R\otimes\mathbb{Q})\langle 1\rangle,$$

and restricting to $BU\langle 2n\rangle$ gives a map

$$c_\beta : BU\langle 2n\rangle \to BO\langle 2n\rangle \to GL_1(R\otimes\mathbb{Q})\langle 1\rangle \approx R\otimes\mathbb{Q}\langle 1\rangle.$$

We use Proposition 3.12 to calculate

$$\begin{aligned}\text{RingSpectra}(MO\langle 2n\rangle, R) &\to \prod_{k\geq n} \pi_{2k}R\otimes\mathbb{Q}\\ \beta &\mapsto (c_{\beta *}v^k)\end{aligned}$$

Let g denote the composition

$$g : (\mathbb{C}P^\infty)^n \xrightarrow{\prod(1-L_i)} BU\langle 2n\rangle \to BO\langle 2n\rangle \xrightarrow{c_\beta} GL_1(R\otimes\mathbb{Q})\langle 1\rangle.$$

Then

$$g = g(x_1,\ldots,x_n) = 1 + \text{higher terms} \in H^0((\mathbb{C}P^\infty)^n_+; R_*\otimes\mathbb{Q}),$$

where x_i is the ordinary cohomology Chern class of L_i. Proposition 3.43 of [AHS01] implies that there is a power series $f(x) = 1 + o(x) \in (R_*\otimes\mathbb{Q})[\![x]\!]$ such that

$$g(x_1,\ldots,x_n) = \prod_{I\subseteq\{1,\ldots,n\}} f(x_I)^{(-1)^{|I|}}.$$

For example, if $n = 3$ then

$$g(x_1,x_2,x_3) = \frac{f(x_1+x_2)f(x_1+x_3)f(x_2+x_3)}{f(x_1+x_2+x_3)f(x_1)f(x_2)f(x_3)}.$$

The Proposition also implies that if f' is another such power series then

$$f'(x) = f(x)\exp(\text{function of the } x_i \text{ of degree } n).$$

It follows that if we write

$$f(x) = \exp\left(\sum_{k\geq 1}\frac{t_k}{k!}x^k\right),$$

then g determines t_k for $k \geq n$. If β factors as

$$MO\langle 2n\rangle \to MSO \xrightarrow{\beta} R,$$

then f may be taken to be the Hirzebruch characteristic series K_β. The argument of Proposition 3.12 then implies

Proposition 3.18. *For $k \geq n$,*

$$c_{\beta *} v^k = (-1)^k t_k. \quad \square$$

$\square$

Remark 3.19. In the case at hand, $c_{\beta *} v^k = 0$ unless $k \equiv 0 \mod 2$, so we could have omitted the sign. The formula is written so that it remains true for complex orientations which do not factor through MSO.

4. Localization of units

In this section and the next, we study the Morava K-theory and E-theory localizations of the spectrum of units of an E_∞ ring spectrum. The main tool is the functor of Bousfield-Kuhn [], which gives rise to the logarithm of [Rez06].

4.1. **The weak equivalence $GL_1 R \approx \Omega^\infty R$.** Let R be an E_∞ ring spectrum. If X is a pointed space, then we have the natural transformation

$$[X, GL_1 R]_* \cong (1 + \tilde{R}^0(X)) \subseteq R^0(X_+), \tag{4.1}$$

which is an isomorphism when X is a connected sphere. It follows that the natural transformation $1 + x \mapsto x$ is represented by a weak equivalence of pointed spaces

$$GL_1 R\langle 1 \rangle \to \Omega^\infty R\langle 1 \rangle. \tag{4.2}$$

4.2. **The Bousfield-Kuhn functor.** Fix a prime p and, for $n \geq 0$ let

$$L_{K(n)}, L^f_{K(n)} : \text{spectra} \to \text{spectra}$$

denote Bousfield localization and finite localization with respect to the indicated Morava K-theory.

The functor $L_{K(n)} gl_1$ is approachable because of the following construction of Bousfield and Kuhn [Bou87, Kuh89]

Theorem 4.3. *For each prime p and each $n \geq 1$, there is a functor*

$$\Phi^f_n : \operatorname{Ho} \text{spaces}_* \to \operatorname{Ho} \text{spectra}$$

and a natural equivalence

$$L^f_{K(n)} \cong \Phi^f_n \Omega^\infty;$$

setting

$$\Phi_n = L_{K(n)} \Phi^f_n,$$

gives a natural equivalence

$$L_{K(n)} \cong \Phi_n \Omega^\infty.$$

$\square$

4.3. **The logarithm.** By applying the Bousfield-Kuhn functor to the weak equivalence (4.2), we obtain weak equivalences

$$\begin{aligned} \ell_n &: L_{K(n)} gl_1 R \approx L_{K(n)} R \\ \ell^f_n &: L^f_{K(n)} gl_1 R \approx L^f_{K(n)} R \end{aligned} \tag{4.4}$$

naturally in the E_∞ spectrum R. The composition

$$gl_1 R \to L_{K(n)} gl_1 R \xrightarrow{\ell_n} L_{K(n)} R$$

represents a "logarithmic" natural transformation

$$\ell_n : (1 + \tilde{R}^0 X)^\times \subseteq R^0(X_+)^\times \to \tilde{R}^0(X).$$

It has been extensively studied by the third author in [Rez06]. In particular, he proves the following.

Proposition 4.5. *If R is a $K(1)$-local E_∞ spectrum, then for $x \in R^0(X_+)^\times$,*

$$\begin{aligned}\ell_1(x) &= \left(1 - \frac{1}{p}\psi\right) \log x \\ &= \frac{1}{p} \log \frac{x^p}{\psi(x)} \\ &= \sum_{k=1}^{\infty} (-1)^k \frac{p^{k-1}}{k} \left(\frac{\theta(x)}{x^p}\right)^k .\end{aligned}$$

□

Example 4.6. For example, take $X = S^{2n}$. Then $R^0(S^{2n}) \cong \pi_0 R[\varepsilon]/\varepsilon^2$, and

$$\begin{aligned}\ell_1(1 + a\varepsilon) &= (1 - \frac{1}{p}\psi) \log(1 + a\varepsilon) \\ &= (1 - \frac{1}{p}\psi)(a\varepsilon).\end{aligned} \tag{4.7}$$

Now suppose that $E = LT(\hat{C}_0)$ is the Lubin-Tate spectrum associated to a supersingular elliptic curve C_0 in characteristic $p > 0$: so E is simultaneously an elliptic spectrum and a form of E_2. Recall that E in an E_∞ ring spectrum.

Let

$$T(p) : E^0(X_+) \to E^0(X_+)$$

be the operation which extends the classical Hecke operator on coefficients. If ψ^p is the power operation associated to the subgroup of p-torsion, let R be the operation

$$R = \frac{1}{p}\psi^p.$$

On $E^0 S^{2n}$, the action of R is given by the formula

$$Ra = p^{n-1} a.$$

Proposition 4.8. *For $x \in E^0(X_+)^\times$,*

$$\ell_2(x) = (1 - T(p) + R) \log x.$$

□

Example 4.9. Taking $f \in \pi_{2n} E$ so $1 + f \in E^0(S^{2n})^\times$, we find that $Rf = p^{n-1} f$, and

$$\ell_2(1 + f) = (1 - T(p) + p^{n-1}) f. \tag{4.10}$$

4.4. **Morava E-theory localization of units.** We write L_n for $L_{K(0)\vee\cdots\vee K(n)}$. It is the localization with respect to the nth Lubin-Tate or Morava E theory. In this section we give a proof of the following result.

Theorem 4.11. *Let R be an E_∞ spectrum such that $R = L_n R$. If F denotes the fiber of the natural map $gl_1 R \to L_n gl_1 R$, then $\pi_* F$ is torsion, and for $q > n$,*

$$\pi_q F = 0.$$

Lemma 4.12. *Let X be a spectrum. Then $L_n^f X \approx L_n X$ if and only $L^f_{K(j)} X \approx L_{K(j)} X$ for $0 \le j \le n$.*

Proof. This follows result follows from the various pull-back squares relating L_n and $L_{K(j)}$ and L_n^f and $L^f_{K(j)}$. □

Suppose that R is an E_∞ spectrum such that $R = L_n R$. Using Lemma 4.12 and the isomorphisms ℓ_j (4.4), we have

$$L_{K(j)} gl_1 R \approx L_{K(j)} R \approx L^f_{K(j)} R \approx L^f_{K(j)} gl_1 R.$$

Applying Lemma 4.12 again, we conclude that

$$L_n^f gl_1 R \approx L_n gl_1 R$$

and so

$$\mathrm{fib}(gl_1 R \to L_n gl_1 R) \approx \mathrm{fib}(gl_1 R \to L_n^f gl_1 R).$$

Let us write F for this fiber. We claim that it is a filtered homotopy colimit of coconnected torsion spectra.

Recall that

$$\mathrm{fib}(S \to L_n^f S) = \operatorname*{hocolim}_{\alpha} Z_\alpha,$$

where Z_α is a filtered colimit of finite complexes of type $n+1$; and if

$$F_\alpha = gl_1 R \wedge Z_\alpha,$$

then

$$F = \operatorname*{hocolim}_{\alpha} F_\alpha.$$

In particular each F_α is torsion.

To see that each F_α is coconnected, let DZ_α be the Spanier-Whitehead dual of Z_α. Let q be large enough that there is a connected finite complex K_α such that

$$\Sigma^q DZ_\alpha = \Sigma^\infty K_\alpha.$$

Then

$$\begin{aligned}
\Omega^\infty \Sigma^{-q} F_\alpha &\approx \Omega^\infty F(\Sigma^q DZ_\alpha, gl_1 R) \\
&\approx \mathrm{spectra}(\Sigma^\infty K_\alpha, gl_1 R) \\
&\approx \mathrm{spaces}_*(K_\alpha, GL_1 R) \\
&\xrightarrow[\approx]{1-x} \mathrm{spaces}_*(K_\alpha, \Omega^\infty R) = *,
\end{aligned}$$

since R is L_n-local and K_α has type $n+1$.

Now we can show that $\pi_i F = 0$ for $i > n$. Let's write P_n for the n Postnikov approximation. Since F_α is torsion and coconnected, we know that

$$\mathrm{fib}(F_\alpha \to P_n F_\alpha)$$

is a homotopy colimit of suspensions of $H\mathbb{F}_p$'s.

Now consider the fibration

$$F \to gl_1 R \to L_n gl_1 R$$

Note that for $q > n \geq j$, $K(\mathbb{F}_p, q)$ is $K(j)$-acyclic by [RW80], and so if $q > n$ then

$$[\Sigma^q H\mathbb{F}_p, gl_1 R] = \pi_0 E_\infty(\Sigma^\infty_+ K(\mathbb{F}_p, q), R) = 0.$$

We also have

$$[\Sigma^q H\mathbb{F}_p, L_n gl_1 R]$$

for all q. It follows that for $q > n$ any map

$$F_\alpha \to F$$

factors through $F_\alpha \to P_n F_\alpha$, and so $F = P_n F$. This completes the proof of Theorem 4.11.

5. String orientations

We apply the obstruction theory in §2 to the study of string orientations. So let $bo = bo\langle 0\rangle$ and bu be the connective real and complex K-theory spectra, and let $bstring = \Sigma^8 bo$, so $\Omega^\infty bstring = BO\langle 8\rangle$ (by Bott periodicity). Let $string = \Sigma^{-1} bstring$, and let j be the map

$$j : string \to gl_1 S$$

which is the desuspension of $bstring \to bgl_1 S$. Let $gl_1 S/string$ be the cofiber of j. Let $MString$ be the homotopy pushout in the diagram of E_∞ spectra

$$\begin{tikzcd} \Sigma^\infty_+\Omega^\infty(gl_1 S) \arrow[r] \arrow[d] & S \arrow[d] \\ \Sigma^\infty_+\Omega^\infty(gl_1 S/string) \arrow[r] & MString. \end{tikzcd} \tag{5.1}$$

Then $MString$ is the Thom spectrum associated to the map

$$BString \to BO;$$

it is also often called $MO\langle 8\rangle$.

Let $\iota : S \to R$ denote the unit of the E_∞ spectrum R, and let $i = gl_1\iota$. The description (5.1) of $MString$, together with the adjunction between $\Sigma^\infty_+\Omega^\infty$ and gl_1, shows that the space $E_\infty(MString, R)$ is naturally weakly equivalent to the homotopy pull-back in the diagram

$$\begin{tikzcd} E_\infty(MString, R) \arrow[r] \arrow[d] & \text{spectra}(gl_1 S/string, gl_1 R) \arrow[d] \\ \{i\} \arrow[r] & \text{spectra}(gl_1 S, gl_1 R). \end{tikzcd} \tag{5.2}$$

This suggests that we make the following

Definition 5.3. If $i : gl_1 S \to X$ is a spectrum under $gl_1 S$, then we write $\mathbf{A}(X)$ for the homotopy pull-back in the diagram

$$\begin{tikzcd} \mathbf{A}(X) \arrow[r] \arrow[d] & \text{spectra}(gl_1 S/string, X) \arrow[d] \\ \{i\} \arrow[r] & \text{spectra}(gl_1 S, X). \end{tikzcd} \tag{5.4}$$

In particular $\mathbf{A}(gl_1 R)$ is naturally weakly equivalent to the space of E_∞ maps $MString \to R$.

In this section we describe a sequence of invariants to detect $\pi_0\mathbf{A}(X)$; they fit into a sequence

$$\pi_0\mathbf{A}(X) \twoheadrightarrow \mathbf{B}(X) \twoheadrightarrow \mathbf{C}(X) \hookrightarrow \mathbf{D}(X).$$

The first approximation, $\mathbf{B}(X)$, arises from the long exact sequence of homotopy groups associated to the square (5.4). By definition,

$$\mathbf{D}(X) \cong [bstring, X \otimes \mathbb{Q}].$$

The map $\mathbf{B}(X) \to \mathbf{D}(X)$ arises from the fact that $gl_1 S$ is rationally contractible, and so it can be calculated using the results of §3. The image of $\pi_0\mathbf{A}(X)$ in $\mathbf{D}(X)$ will be called $\mathbf{C}(X)$.

5.1. **The homotopy invariant $\mathbf{B}(X)$.**

Consider the diagram

$$\begin{tikzcd} string \arrow[r, "j"] & gl_1 S \arrow[r, "\pi"] \arrow[d, "i"'] & gl_1 S/string \arrow[r] \arrow[dl, dashed, "u"] & bstring \\ & X & & \end{tikzcd} \tag{5.5}$$

Let

$$\mathbf{B}(X) \overset{\text{def}}{=} \{u \in [gl_1 S/string, X] | u\pi = i\},$$

be the set of dotted arrows in the homotopy category making the diagram commute. The long exact sequence of homotopy groups associated to the diagram (5.4) includes a natural surjective map

$$h : \pi_0 \mathbf{A}(X) \twoheadrightarrow \mathbf{B}(X). \tag{5.6}$$

Note also that the map

$$[bstring, X] \to [gl_1 S/string, X]$$

induces an action of $[bstring, X]$ on $\pi_0 \mathbf{A}(X)$ and on $\mathbf{B}(X)$.

Lemma 5.7. *The map h is compatible with the action of $[bstring, X]$ on its source and target. If $\mathbf{A}(X)$ is non-empty, then it is weakly equivalent to* spectra$(bstring, X)$*, and $\pi_0 \mathbf{A}(X)$ is a torsor for $[bstring, X]$.*

Proof. The compatibility of h with the action of $[bstring, X]$ is a tautology. The rest is follows from taking $U = gl_1 S$, $V = gl_1 S/string$, and $X = X$ in Lemma 2.12. □

Example 5.8. Taking $X = gl_1 R$ in the Lemma, we recover the fact that if $E_\infty(MString, R)$ is non-empty, then $\pi_0 E_\infty(MString, R)$ is a torsor for

$$\pi_0 E_\infty(\Sigma^\infty_+ BString, R) \cong [bstring, gl_1 R].$$

5.2. **The characteristic series and the Miller invariant.**

Since, by Corollary 3.5, $(gl_1 S) \otimes \mathbb{Q}$ is contractible, we have weak equivalences

$$\mathbf{A}(X \otimes \mathbb{Q}) \approx \text{spectra}(gl_1 S/string, X \otimes \mathbb{Q}) \approx \text{spectra}(bstring, X \otimes \mathbb{Q}).$$

Definition 5.9. If X is a spectrum (or a pointed space), let

$$\mathbf{D}(X) \overset{\text{def}}{=} \left\{ s \in \prod_{k \geq 4} \pi_{2k} X \otimes \mathbb{Q} \;\middle|\; s_k = 0 \text{ if } k \text{ is odd} \right\}.$$

If X is a spectrum, then there is a natural isomorphism

$$s : [bstring, X \otimes \mathbb{Q}] \cong \mathbf{D}(X) \tag{5.10}$$

sending a map $f : bstring \to X \otimes \mathbb{Q}$ to the sequence $s(f)$ defined by

$$\text{“}s(f)_k = f_* v^k\text{,”}$$

where we identify $\pi_* bstring$ with its image in $\pi_* bu = \mathbb{Z}[v]$ under complexification. Precisely, $s(f)_k$ is defined so that, for $x \in \pi_{2k} bstring$,

$$f_* x = \lambda \cdot s(f)_k$$

if $c_* x = \lambda \cdot v^k$, where $c : bstring \to bu$ is induced by complexification.

Definition 5.11. If $i : gl_1 S \to X$ is a spectrum under $gl_1 S$, then the *characteristic map* of X is the map

$$b : \pi_0 \mathbf{A}(X) \to \mathbf{D}(X)$$

given by the composition

$$b : \pi_0 \mathbf{A}(X) \to \pi_0 \mathbf{A}(X_{\mathbb{Q}}) \cong [bstring, X_{\mathbb{Q}}] \underset{\cong}{\xrightarrow{s}} \mathbf{D}(X). \tag{5.12}$$

We write

$$\mathbf{C}(X) \overset{\text{def}}{=} (\operatorname{im} b : \pi_0 \mathbf{A}(X) \to \mathbf{D}(X)) \subseteq \mathbf{D}(X)$$

for the image of the characteristic map.

If R is an E_∞ spectrum, then we may write the charactierstic map of $gl_1 R$ as

$$b : \pi_0 E_\infty(MString, R) \to \mathbf{D}(gl_1 R),$$

and the methods of §3.3 lead to an expression of the characteristic map using Hirzebruch's theory of multiplicative sequences. Before giving the formula, we note that the characterstic map is the refinement of an unstable invariant.

By Proposition 2.10, the standard orientation

$$MString \to MSO \to H\mathbb{Q} \approx S \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$$

corresponds to a section

$$\begin{tikzcd} & EGL_1 R \otimes \mathbb{Q} \arrow[d] \\ BString \arrow[ur, "\alpha"] \arrow[r] & BGL_1 R \otimes \mathbb{Q}, \end{tikzcd}$$

while a (not-necessarily E_∞) orientation

$$\beta : MString \to R$$

gives a section

$$\begin{tikzcd} & EGL_1 R \arrow[d] \\ BString \arrow[ur, "\beta"] \arrow[r] & BGL_1 R. \end{tikzcd}$$

The difference of α and β is a map

$$\Delta = \delta(\alpha, \beta) \in [BString, GL_1 R \otimes \mathbb{Q}],$$

and we define

$$b(\beta) \overset{\text{def}}{=} (\Delta_* v^k)_{k \geq 4} \in \mathbf{D}\,(GL_1 R)\,.$$

The notation is consistent: if β is an E_∞ map, then Proposition 2.10 gives

$$\delta \in [bstring, gl_1 R \otimes \mathbb{Q}],$$

such that

$$\Delta = \Omega^\infty \delta$$

and

$$(\delta_* v^k)_{k \geq 4} = b(\beta) \in \mathbf{D}\,(gl_1 R)\,,$$

so the two notions of $b(\beta)$ coincide in

$$\mathbf{D}\,(gl_1 R) \cong \mathbf{D}\,(GL_1 R)\,.$$

In any case, as long as β is merely homotopy-multiplicative, there is the following calculation. View the composition

$$(\mathbb{C}P^\infty)^3 \xrightarrow{\prod(1-L_i)} BU\langle 6\rangle \xrightarrow{r} BString \xrightarrow{\delta(\alpha,\beta)} GL_1 R \otimes \mathbb{Q}$$

to get an element

$$g = g(x_1, x_2, x_3) = 1 + o(x_1 x_2 x_3) \in H^0((\mathbb{C}P^\infty)^3_+; R_* \otimes \mathbb{Q})^\times.$$

As we explain in §3.3, there is a power series $h(x) = R_* \otimes \mathbb{Q}[\![x]\!]$ such that

$$g(x_1, x_2, x_3) = \prod_{I \subseteq \{1,2,3\}} h(x_I)^{(-1)^{|I|}}; \tag{5.13}$$

this equation does not quite determine h, but if we write

$$h(x) = \exp\left(\sum_{k \geq 1} t_k \frac{x^k}{k!}\right),$$

then (5.13) determines $t_k = t_k(\beta)$ for $k \geq 3$. If β factors rationally through an orientation

$$\gamma : MSO \to R \otimes \mathbb{Q},$$

then we may take h to be the Hirzebruch series

$$h(x) = K_\gamma(x) = \frac{x}{\exp_{F_\beta}(x)}$$

of γ.

Proposition 5.14. *With the definitions above,*

$$t_k(\beta) = 2b_k(\beta)$$

for $k \geq 4$, *and so*

$$h(x) = \exp\left(2\sum b_k(\beta)\frac{x^k}{k!}\right).$$

Proof. In Proposition 3.18, it is shown that

$$\delta_* r_* v^k = (-1)^k t_k = t_k \in \pi_{2k}R \otimes \mathbb{Q}$$

(using the fact that $t_k = 0$ unless k is even). Since

$$c_* r_* v^k = v^k + (-1)^k v^k,$$

we find that $t_k = 2b_k$, as required. □

Example 5.15. The Hirzebruch series of the Atiyah-Bott-Shapiro orientation

$$ABS : MSpin \to KO$$

is

$$\frac{x}{e^{x/2} - e^{-x/2}} = \exp\left(-\sum_{k\geq 2}\frac{B_k}{k}\frac{x^k}{k!}\right), \tag{5.16}$$

where B_k is the k^{th} Bernoulli number (see Proposition 10.2). It follows that the characteristic map of the Atiyah-Bott-Shapiro orientation is given by

$$b_k(ABS) = -\frac{B_k}{2k}v^k \in \pi_{2k}KO \otimes \mathbb{Q}.$$

Note that the map

$$BString \xrightarrow{\delta(\alpha,\beta)} GL_1R \otimes \mathbb{Q} \to GL_1R \otimes \mathbb{Q}/\mathbb{Z}$$

is independent of the choice of orientation β; it is the Miller invariant $M(BString, GL_1R)$ as in §3.5. Since we are fixing the spectrum *bstring* as our source, we make the following abbreviation.

Definition 5.17. Let $i : gl_1S \to X$ be a spectrum under gl_1S. We write $\mathbf{m}_X$ for the stable Miller invariant $m(bstring, X)$. Similarly if X is a space under GL_1S, we write $\mathbf{M}_X$ for the unstable invariant $M(BString, X)$. We may write $\mathbf{M}_R$ for $\mathbf{M}_{GL_1R}$ and $\mathbf{m}_R$ for $\mathbf{m}_{gl_1R}$ where appropriate.

Of course these are related by the formula

$$\mathbf{M}_{\Omega^\infty X} = \Omega^\infty \mathbf{m}_X,$$

and so equation (5.16) implies the following.

Proposition 5.18.

$$(\mathbf{m}_{KO})_* v^k = -\frac{B_k}{2k}v^k \mod \mathbb{Z}$$

in $\pi_{2k}KO \otimes \mathbb{Q}/\mathbb{Z}$ *(with the convention that* $v^k = 0$ *for* k *odd, or noting that* $B_k = 0$ *for* k *odd and bigger than* 1*).* □

The following is a useful summary of the relationship among our invariants.

Proposition 5.19. i) *The map* $b : \pi_0\mathbf{A}(X) \to \mathbf{D}(X)$ *factors through the map* h *of* (5.6)*, and so we have the sequence of epi- and monomorphisms*

$$\pi_0\mathbf{A}(X) \twoheadrightarrow \mathbf{B}(X) \twoheadrightarrow \mathbf{C}(X) \rightarrowtail \mathbf{D}(X).$$

ii) *If* $\alpha, \beta \in \pi_0\mathbf{A}(X)$ *are such that* $b(\alpha) = b(\beta)$*, then they differ by a torsion element of* $[bstring, X]$.

iii) *If* $\mathbf{A}(X)$ *is non-empty, then there is a "short exact sequence"*

$$0 \to [bstring, X]_{tors} \to \pi_0\mathbf{A}(X) \twoheadrightarrow \mathbf{C}(X).$$

iv) *If $\mathbf{C}(X)$ is non-empty, then it is the set of*

$$f \in [bstring, X \otimes \mathbb{Q}]$$

such that

$$bstring \xrightarrow{f} X \otimes \mathbb{Q} \to X \otimes \mathbb{Q}/\mathbb{Z}$$

is $\mathbf{m}_X$.

v) *The functors $\mathbf{A}(-)$ and $\mathbf{A}(gl_1 -)$ preserve homotopy limits.*

vi) *If $g : X \to Y$ is a map of spectra under $gl_1 S$ such that*

$$\pi_q \operatorname{fib}(g) = 0$$

for $q \geq 6$, then

$$\begin{aligned}
\mathbf{A}(X) &\approx \mathbf{A}(Y) \\
\mathbf{B}(X) &\cong \mathbf{B}(Y) \\
\mathbf{D}(X) &\cong \mathbf{D}(Y) \\
\mathbf{C}(X) &\cong \mathbf{C}(Y).
\end{aligned}$$

Proof. Most of this is clear from the definitions. For items (i) and (iv) it may be helpful to contemplate the diagram

$$\begin{tikzcd}
gl_1 S \arrow[r] \arrow[d] & gl_1 S/string \arrow[r] \arrow[d, dashed] & bstring \arrow[r, "\Sigma j"] \arrow[d, dashed] \arrow[dr, "\mathbf{m}_X"] & bgl_1 S \arrow[d] \\
\Sigma^{-1} X \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] & X \arrow[r] & X \otimes \mathbb{Q} \arrow[r] & X \otimes \mathbb{Q}/\mathbb{Z},
\end{tikzcd}$$

whose rows are cofiber sequences.

For item (vi), note that the hypotheses imply that

$$\operatorname{spectra}(string, \operatorname{fib}(g))$$

is contractible. Consider the diagram

$$\begin{tikzcd}
string \arrow[r, "j"] \arrow[dr, "k"'] & gl_1 S \arrow[r] \arrow[d, "i"] & gl_1 S/string \arrow[dl, dashed] \arrow[d, dashed] \\
\operatorname{fib}(g) \arrow[r] & X \arrow[r, "g"] & Y
\end{tikzcd}$$

If $\mathbf{A}(Y)$ is nonempty, then gk is null, and then $\pi_q \operatorname{fib}(g) = 0$ for $q \leq 6$ implies that k is null, and so $\mathbf{A}(X)$ is nonempty. Thus the hypothesis on $\operatorname{fib}(g)$ implies that $\mathbf{A}(Y)$ is nonempty if and only if $\mathbf{A}(X)$ is nonempty, and in that case we have

$$\mathbf{A}(X) \approx \operatorname{spectra}(bstring, X) \approx \operatorname{spectra}(bstring, Y) \approx \mathbf{A}(Y),$$

using Lemma 2.12. □

6. String orientations of KO

Over the next few section we assemble a proof of the following.

Theorem 6.1. *The characteristic map b identifies*

$$\pi_0 \mathbf{A}(gl_1 KO) \cong \pi_0 E_\infty(MString, KO) \cong \pi_0 E_\infty(MSpin, KO)$$

with the set of even sequences $\{b_k \in \mathbb{Q}\}_{k \geq 4}$ satisfying the following conditions.

i) $b_k \equiv -\frac{B_k}{2k} \mod \mathbb{Z}$.

ii) *For each prime p and each $c \in \mathbb{Z}_p^\times$, the sequence $\{(1 - c^k)(1 - p^{k-1}) b_k\}_{k \geq 4}$ satisfies the* generalized Kummer congruences *(Definition 9.6).*

The sequence $\{b_k = -\frac{B_k}{2k}\}_{k\geq 4}$ satisfies these conditions, and so $\pi_0 E_\infty(MSpin, KO)$ is nonempty. The E_∞ orientation with characteristic series $\{b_k\}_{k\geq 4}$ refines the Atiyah-Bott-Shapiro orientation.

Remark 6.2. Michael Joachim [Joa01] has shown that the Atiyah-Bott-Shapiro orientation is an E_∞ map, and Laures [Lau03] has proved this result for 2-adic real K-theory.

We first describe the string orientations of p-adic real K-theory in §7. In §8 we explain how our description of the p-adic orientations fit together to describe integral orientations. Sections 9 and 10 give proofs of technical results which were used along the way.

7. String orientations of KO_p

We begin with the study of string orientations of p-adic real K-theory. To begin, we recall that KO_p is $K(1)$-local.

Lemma 7.1. *The natural map*

$$bstring \to KO_p$$

is $K(1)$-localization:

$$L_{K(1)} bstring \approx KO_p.$$

Proof. Recall that K_p is $K(1)$-local, and indeed as bu is a BU-module spectrum,

$$L_{K(1)} bu = (v_1^{-1} bu)^\wedge_p = K_p.$$

The fibration

$$KO_p \to K_p \to K_p$$

then shows that KO_p is $K(1)$-local. Next, observe that $L_{K(1)} bstring \approx L_{K(1)} bo$, and the result of Bousfield-Kuhn (Theorem 4.3) implies that

$$L_{K(1)} bstring \approx L_{K(1)} KO_p \approx KO_p.$$

□

In view the the Lemma, (4.4) specializes to give the logarithmic weak equivalence

$$\ell_1 : L_{K(1)} gl_1 KO_p \to L_{K(1)} KO_p \approx KO_p \tag{7.2}$$

Lemma 7.3. *The natural map*

$$gl_1 KO_p \to L_{K(1)} gl_1 KO_p \xrightarrow[\approx]{\ell_1} L_{K(1)} KO_p \approx KO_p$$

induces a weak equivalence

$$\mathbf{A}(gl_1 KO_p) \approx \mathbf{A}(KO_p); \tag{7.4}$$

and in the diagram

$$\begin{array}{ccccc} \pi_0 \mathbf{A}(gl_1 KO_p) & \longrightarrow & \mathbf{B}(gl_1 KO_p) & \longrightarrow & \mathbf{C}(gl_1 KO_p) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_0 \mathbf{A}(KO_p) & \longrightarrow & \mathbf{B}(KO_p) & \longrightarrow & \mathbf{C}(KO_p), \end{array} \tag{7.5}$$

all the arrows are isomorphisms.

Proof. KO_p is $K(1)$-local and so L_1-local; it follows from Theorem 4.11 that

$$\pi_q \operatorname{fib}(gl_1 KO_p \to L_{K(1)} gl_1 KO_p) = 0$$

for $q > 1$, and so

$$\mathbf{A}(gl_1 KO_p) \cong \mathbf{A}\left(L_{K(1)} gl_1 KO_p\right)$$

by Proposition 5.19. The weak equivalence (7.2) then gives the weak equivalence (7.4). Similar remarks apply for the the vertical arrows in (7.5).

For the horizontal arrows, recall also from Proposition 5.19 that we always have surjections

$$\pi_0 \mathbf{A}(X) \twoheadrightarrow \mathbf{B}(X) \twoheadrightarrow \mathbf{C}(X),$$

and the kernel consists of torsion elements of $[bstring, X]$. We have (using Lemma 7.1 along the way)

$$[bstring, gl_1 KO_p] \cong [bstring, L_{K(1)} gl_1 KO_p] \cong [bstring, KO_p] \cong [KO_p, KO_p],$$

which is torsion-free. □

Corollary 7.6. *$\pi_0 E_\infty(MString, KO_p) \cong \pi_0 \mathbf{A}(gl_1 KO_p) \cong \mathbf{B}(gl_1 KO_p) \cong \mathbf{B}(KO_p)$ is non-empty.*

Proof. Consider the diagram

$$\begin{tikzcd} string \arrow[r] & gl_1 S \arrow[r] \arrow[d] & gl_1 S/string \arrow[dl, dashed] \\ & KO_p, & \end{tikzcd}$$

where the vertical arrow is

$$gl_1 S \to gl_1 KO_p \xrightarrow{\ell_1} KO_p.$$

Since KO_p is $K(1)$-local and $L_{K(1)} string = \Sigma^{-1} KO_p$,

$$[string, KO_p] = [\Sigma^{-1} KO_p, KO_p] = 0.$$

Thus a dotted arrow exists, and $\mathbf{B}(KO_p)$ is nonempty. □

To study $\mathbf{B}(KO_p)$, let c be a p-adic unit. Let j_c be the cofiber in

$$string \xrightarrow{\Sigma^{-1}(1-\psi^c)} string \to j_c,$$

and let $J_c = \Omega^\infty j_c$. The solution to the Unstable Adams Conjecture gives maps A_c and B_c making the diagram

$$\begin{tikzcd} J_c \arrow[r] \arrow[d, "A_c"'] & BString \arrow[r, "\Omega^\infty(1-\psi^c)"] \arrow[d, "B_c"'] & BString \arrow[d, equal] & \\ GL_1 S \arrow[r] & GL_1 S/String \arrow[r] & BString \arrow[r, "\Omega^\infty j"] & BGL_1 S \end{tikzcd} \tag{7.7}$$

commute. Applying the Bousfield-Kuhn functor Φ (4.3) to the diagram (7.7) gives the top portion of the commutative diagram

$$\begin{tikzcd} L_{K(1)} j_c \arrow[r] \arrow[d, "\Phi A_c"', "\approx"] & KO_p \arrow[r, "1-\psi^c"] \arrow[d, "\Phi B_c"', "\approx"] & KO_p \arrow[d, equal] & \\ L_{K(1)} gl_1 S \arrow[r] \arrow[d] & L_{K(1)} gl_1 S/string \arrow[r] \arrow[d, dashed, "\alpha"'] & KO_p \arrow[r] \arrow[d, dashed, "\beta"'] \arrow[dr, "\mathbf{m}_{gl_1 KO_p}"] & L_{K(1)} bgl_1 S \arrow[d] \\ \Sigma^{-1} L_{K(1)} gl_1 KO_p \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "\ell_1"'] & L_{K(1)} gl_1 KO_p \arrow[r] \arrow[d, "\ell_1"', "\approx"] & L_{K(1)} gl_1 KO_p \otimes \mathbb{Q} \arrow[r, "r"'] \arrow[d, "\ell_1"', "\approx"] & L_{K(1)} gl_1 KO_p \otimes \mathbb{Q}/\mathbb{Z} \arrow[d, "\ell_1"', "\approx"] \\ \Sigma^{-1} KO_p \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] & KO_p \arrow[r] & KO_p \otimes \mathbb{Q} \arrow[r] & KO_p \otimes \mathbb{Q}/\mathbb{Z} \end{tikzcd} \tag{7.8}$$

in which the rows are cofibrations.

Lemma 7.9. *If c is a generator of $\mathbb{Z}_p^\times/\{\pm 1\}$, then the map ΦB_c is a weak equivalence.*

Proof. The assertion is equivalent to the assertion that ΦA_c is a weak equivalence. Thus the result is an expression of the calculation of the $K(1)$-local sphere [Bou79, Rav84]. □

Proposition 7.10. *Composition with ΦB_c and with ℓ_1 in diagram (7.8) identifies the set of $\alpha \in \mathbf{B}\left(L_{K(1)} gl_1 KO_p\right) \cong \pi_0 E_\infty(MString, KO_p)$ with the set of sequences $b \in \mathbf{D}\left(KO_p\right) \subseteq \prod_{k\geq 4} \mathbb{Q}_p$ such that*

i) *the sequence $\{(1-c^k)(1-p^{k-1})b_k\}_{k\geq 4}$ satisfies the* generalized Kummer congruences*; and*

ii) $b_k \equiv -\frac{B_k}{2k} \mod \mathbb{Z}_p$.

Proof. First, it is clear from the diagram that to give a map

$$\alpha : L_{K(1)} gl_1 S/string \to L_{K(1)} gl_1 KO_p \tag{7.11}$$

is equivalent to giving the element $\ell_1 \alpha \Phi B_c$ of $[KO_p, KO_p]$. By Proposition 9.7, applying π_* as in

$$[KO_p, KO_p] \to \prod_{k \text{ even } \geq 4} \pi_{2k} KO_p \to \prod_{k \text{ even } \geq 4} \mathbb{Q}_p$$

is an isomorphism onto the set of even sequences satisfying the generalized Kummer congruences. Thus to give a map α as in (7.11) is equivalent to giving a sequence

$$\{t_k(\alpha) = \pi_{2k}(\ell_1 \alpha \Phi B_c)\}_{k\geq 4} \in \prod \mathbb{Q}_p$$

satisfying the generalized Kummer congruences.

By definition, $b(\alpha)$ is the sequence

$$\{b_k(\alpha) = \pi_{2k}\beta\} \in \prod_{k\geq 4} \pi_{2k} gl_1 KO_p \otimes \mathbb{Q} \cong [bstring, gl_1 KO_p \otimes \mathbb{Q}]$$

where β is the map induced by α in the diagram (7.8). To compare $t(\alpha)$ and $b(\alpha)$, it is convenient first to consider

$$b(\ell_1 \alpha) \in \mathbf{D}\left(KO_p\right) \cong [bstring, KO_p \otimes \mathbb{Q}].$$

Rezk's formula (4.6) shows that

$$b_k(\ell_1 \alpha) = (1-p^{k-1}) b_k(\alpha).$$

Inspection of diagram (7.8) shows that

$$t_k(\alpha) = (1-c^k)(1-p^{k-1}) b_k(\alpha).$$

Thus to give a pair of maps α and β making the middle square in (7.8) commute is equivalent to giving a sequence $\{b_k(\alpha)\} \in \mathbf{D}\left(KO_p\right) \subset \prod_{k\geq 4} \mathbb{Q}_p$ such that the sequence $\{(1-c^k)(1-p^{k-1})b_k(\alpha)\}$ satisfies the generalized Kummer congruences.

Such α, β make the whole diagram (7.8) commute, and so correspond to an E_∞ orientation $MString \to KO_p$, if and only if

$$r\beta = \mathbf{m}_{gl_1 KO_p},$$

where r is the map $gl_1 KO_p \otimes \mathbb{Q} \to gl_1 KO_p \otimes \mathbb{Q}/\mathbb{Z}$. This equation holds if and only if it holds after applying π_*. Using Proposition 5.18 and the definition of $b_k(\alpha)$, this is the condition that

$$b_k(\alpha) = -\frac{B_k}{2k} \mod \mathbb{Z}_p$$

for $k \geq 4$. □

Corollary 7.12. *There is a unique E_∞ map*

$$MO\langle 8\rangle \to KO_p$$

refining the Atiyah-Bott-Shapiro orientation.

Proof. According to the preceding Proposition, the statement is equivalent to the fact that the sequence

$$\{-(1-p^{k-1})(1-c^k)\frac{B_k}{2k}\}_{k\geq 4}$$

satisfies the generalized Kummer congruences. This is equivalent to the existence of the Mazur measure; see Example 9.9 and Corollary 9.10. □

We conclude this section by describing another approach to the commutativity of the diagram (7.8) which will be useful in the study of tmf orientations. Consider the diagram

$$\begin{tikzcd}
L_{K(1)}S \arrow[r, "\rho(c)"] & KO_p \arrow[d, "\Phi B_c"', "\approx"] \\
L_{K(1)}gl_1S \arrow[u, "\ell_1", "\approx"'] \arrow[r] \arrow[d] \arrow[dr, "L_{K(1)}gl_1\iota"'] & L_{K(1)}gl_1S/string \arrow[d, dashed, "\alpha"'] \\
\Sigma^{-1}L_{K(1)}gl_1KO_p \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d] & L_{K(1)}gl_1KO_p \arrow[d, "\ell_1"', "\approx"] \\
\Sigma^{-1}KO_p \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] & KO_p,
\end{tikzcd} \tag{7.13}$$

which except for the top square is a fragment of (7.8). The map $\rho(c)$ is the map in the homotopy category defined so that the top square commutes.

Suppose given *any* map

$$\alpha : L_{K(1)}gl_1S/string \to L_{K(1)}gl_1KOp.$$

In view of the equivalences marked in the diagram, the middle square commutes if and only if

$$\ell_1\alpha\Phi B_c\rho(c) = \ell_1 L_{K(1)}gl_1\iota\ell_1^{-1} \in \pi_0KO_p \cong \mathbb{Z}_p.$$

The naturality of ℓ_1 implies

Lemma 7.14. *For any E_∞ spectrum R,*

$$\ell_1 L_{K(1)}gl_1\iota\ell_1^{-1} = L_{K(1)}\iota : L_{K(1)}S \to L_{K(1)}R.$$

□

Proposition 7.15.

$$\rho(c)^{-1} = \frac{1}{2p}\log(c^{p-1}) \in \pi_0KO_p.$$

Proof. Let α be chosen so that the diagram (7.8) (and so also (7.13)) commutes. Let

$$g = \ell_1\alpha\Phi B_c : KO_p \to KO_p,$$

and define $b_k \in \mathbb{Q}_p$ by the formula

$$g_*v^k \equiv (1-c^k)(1-p^{k-1})b_kv^k.$$

From Proposition 7.10 we know that

$$b_k \equiv -\frac{B_k}{2k} \mod \mathbb{Z}. \tag{7.16}$$

By the definition of $\rho(c)$ we have

$$g\rho(c) = \ell_1L_{K(1)}gl_1\iota\ell_1^{-1} \in \pi_0KO_p \cong \mathbb{Z}_p,$$

and from Lemma 7.14 we have $\ell_1L_{K(1)}gl_1\iota\ell_1^{-1} = L_{K(1)}\iota$. Thus

$$g\rho(c) = 1,$$

and it remains to calculate $\pi_0 g$.

As we explain in Proposition 9.7, if for k even and ≥ 4, $g_k \in \mathbb{Z}_p$ is defined by

$$g_*v^k = g_kv^k$$

then

$$\pi_0 g = \lim_{r\to\infty} g_{(p-1)p^r}.$$

In Corollary 10.7, we show that if b_k is any sequence of p-adic numbers satisfying (7.16), and if $g_k = (1-c^k)(1-p^{k-1})b_k$, then

$$\lim_{r\to\infty} g_{(p-1)p^r} = \frac{1}{2p}\log c^{p-1}.$$

$\square$

8. Proof of Theorem 6.1

When we assemble the orientations of KO_p constructed in §7 into orientations of KO, the result is a proof of Theorem 6.1.

Proposition 5.19 and Lemma 7.3 imply the following.

Lemma 8.1. *Applying $\mathbf{A}\,(gl_1(\,-\,))$ to the homotopy pull-back square*

$$\begin{array}{ccc} KO & \longrightarrow & \prod_p KO_p \\ \downarrow & & \downarrow \\ KO\otimes\mathbb{Q} & \longrightarrow & \left(\prod_p KO_p\right)\otimes\mathbb{Q} \end{array}$$

yields a Cartesian square

$$\begin{array}{ccc} \pi_0\mathbf{A}\,(gl_1 KO) & \longrightarrow & \prod_p \pi_0\mathbf{A}\,(gl_1 KO_p) \\ \downarrow & & \downarrow \\ \mathbf{D}\,(gl_1 KO) & \longrightarrow & \prod_p \mathbf{D}\,(gl_1 KO_p)\,. \end{array}$$

$\square$

Proof of Theorem 6.1. By Lemma 8.1, the characteristic map b identifies $\pi_0 E_\infty(MString, KO)$ with the subset of sequences $\{b_k\}_{k\geq 4} \in \mathbf{D}\,(gl_1 KO) \subseteq \prod\mathbb{Q}$ such that, for each p, $\{b_k\}_{k\geq 4}$ is in $\mathbf{C}\,(gl_1 KO_p)$. By Proposition 7.10, this is the set of $\{b_k\}_{k\geq 4}\in\mathbf{D}\,(gl_1 KO)$ such that, for each prime p and each p-adic unit c, the sequence $\{(1-c^k)(1-p^{k-1})b_k\}_{k\geq 4}$ satisfies the generalized Kummer congruences, and such that

$$b_k \equiv -\frac{B_k}{2k} \mod \mathbb{Z}.$$

In Corollary 7.12, we observe that the charateristic map of the Atiyah-Bott-Shapiro orientation is

$$b_k = -\frac{B_k}{2k},$$

which satisfies all the required, and so corresponds to a unique E_∞ map $MString \to KO$. $\square$

9. Maps between p-adic K-theory spectra

In this section, we fix a prime p. If E denotes either K_p or KO_p, then we write $E^\vee X$ for the "completed homology"

$$E^\vee X \overset{\text{def}}{=} \pi_0 L_{K(1)} E\wedge X.$$

The following results are well-known to experts, and proofs of many cases are available; see for example [Rav84]. What we need can be deduced from [AHS71]. To state a result, note that given

$$f : S \to L_{K(1)}K_p \wedge K_p$$

and $\lambda\in\mathbb{Z}_p^\times$, we get an element

$$S \xrightarrow{f} L_{K(1)}K_p\wedge K_p \xrightarrow{1\wedge\psi^\lambda} L_{K(1)}K_p\wedge K_p \to K_p \in \pi_0 K \cong \mathbb{Z}_p \tag{9.1}$$

(recalling that K_p is $K(1)$-local). Fixing f and letting λ vary over $\mathbb{Z}_p^\times$, we see that f defines a continuous map $\mathbb{Z}_p^\times \to \mathbb{Z}_p$.

Proposition 9.2. *The procedure above induces isomorphisms*

$$
\begin{aligned}
K_p^\vee K_p &\cong \mathrm{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p)\\
KO_p^\vee KO_p &\cong \mathrm{cts}(\mathbb{Z}_p^\times/\{\pm 1\}, \mathbb{Z}_p).
\end{aligned}
\tag{9.3}
$$

Dually, one has

$$
\begin{aligned}
K_p^0 K_p &\cong \hom_{cts}(\mathrm{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p), \mathbb{Z}_p)\\
KO_p^0 KO_p &\cong \hom_{cts}(\mathrm{cts}(\mathbb{Z}_p^\times/\{\pm 1\}, \mathbb{Z}_p), \mathbb{Z}_p).
\end{aligned}
$$

Proof. The statements about cohomology follow from the ones about completed homology, by duality. For the statement about $K_p^\vee K_p$, recall that [AHS71] show that

$$K_0 K \cong \{f \in \mathbb{Q}[x, x^{-1}] | f(k) \in \mathbb{Z}[1/k] \text{ for all } k\};$$

where given

$$f : S \to K \wedge K,$$

$f(k)$ is the composition

$$f(k) : S \xrightarrow{f} K \wedge K \xrightarrow{1 \wedge \psi^k} K \wedge K[\frac{1}{k}] \to K[\frac{1}{k}].$$

Let c be a p-adic unit. For every $r > 0$, there is an integer k prime to p such that $c \equiv k \mod p^r$. Given $f \in K_0 K$ as above, we can consider the class of $f(k) \in \mathbb{Z}[1/k]/p^r = \mathbb{Z}/p^r$. This class depends only on the class of c in $(\mathbb{Z}/p^r)^\times$ and on the class of f in $(K/p^r)_0 K$. Thus we have defined a map

$$(K/p^r)_0 K \to \mathrm{map}(\mathbb{Z}_p^\times, \mathbb{Z}/p^r)$$

which passes to the limit to give (9.3). The case of KO is similar, using the calculation of [AHS71] that

$$KO_0 KO \cong \{f \in \mathbb{Q}[x, x^{-1}] | f(-x) = f(x); f(k) \in \mathbb{Z}[1/k] \text{ for all } k.\}$$

□

Thus $K_p^0 K_p$ is the space of Zp-valued *measures* on $\mathrm{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$. We write $d\mu$ for $\mu \in K^0 K$ viewed as a measure, and for

$$f \in \mathrm{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$$

we use the notation

$$\int f d\mu \overset{\text{def}}{=} \langle \mu, f \rangle.$$

Example 9.4. The formula (9.1) shows that, for $\lambda \in \mathbb{Z}_p^\times$,

$$\int f d\psi^\lambda = f(\lambda)$$

so $d\psi^\lambda$ is the Dirac measure supported at λ.

Example 9.5. In the other direction, if we write x^k for the function

$$(x \mapsto x^k) \in \mathrm{cts}(\mathbb{Z}_p^\times, \mathbb{Z}_p),$$

then

$$\int x^k d\psi^\lambda = \lambda^k = \pi_{2k} \psi^\lambda,$$

and it follows that for $\alpha \in K_p^0 K_p$,

$$\pi_{2k} \alpha = \int x^k d\alpha : \mathbb{Z}_p \to \mathbb{Z}_p.$$

That is, the effect of α viewed is a map of spectra is given by the moments of the measure $d\alpha$.

We shall need to identify those sequences of p-adic numbers which are the moments of measures; equivalently, we shall need to identify which sequences of p-adic numbers are the effect in homotopy of a self-map of KO_p.

Fix $n \geq 0$. Let A_n be the set of polynomials

$$h(x) = \sum_{k \geq n} a_k x^k \in \mathbb{Q}_p[x]$$

such that

$$h(c) \in \mathbb{Z}_p$$

if $c \in \mathbb{Z}_p^\times$.

Definition 9.6. We say that a sequence $\{z_k\}_{k \geq n}$ satisfies the *generalized Kummer congruences* if, for all

$$h(x) = \sum a_k x^k \in A_n,$$

we have

$$\sum a_k z_k \in \mathbb{Z}_p.$$

Proposition 9.7. *Let n be a natural number. The natural map*

$$K_p^0 K_p \to \prod_{k \geq n} \mathbb{Q}_p$$

$$\alpha \mapsto \{\pi_{2k}\alpha\}_{k \geq n}$$

is injective. Its image is the set of sequences $\{z_k\}$ satisfying the generalized Kummer congruences. For any such sequence $\{z_k\}$, the limit

$$z_0 = \lim_r z_{(p-1)p^r}$$

exists, and if $z_k = \pi_{2k}\alpha$ for $k \geq n$, then $\pi_0\alpha = z_0$. Similarly, the natural map

$$KO_p^0 KO_p \xrightarrow{s} \prod_{k \geq n} \mathbb{Q}_p$$

$$\alpha \mapsto \{\pi_{2k}\alpha\}_{k \geq n}$$

is injective, with image the set of sequences $\{z_k\}$ with $z_k = 0$ for k odd, and satisfying the generalized Kummer congruences.

Proof. A polynomial $h = \sum a_k x^k \in A_n$ satisfying the conditions of the proposition defines a continuous function

$$h : \mathbb{Z}_p^\times \to \mathbb{Z}_p.$$

Suppose $\alpha \in K^0 K$, and let

$$z_k = \pi_{2k}\alpha = \int x^k d\alpha.$$

for k even. Then

$$\mathbb{Z}_p \ni \int h d\alpha = \sum_k a_k z_k$$

So the sequence $\{\pi_{2k}\alpha\}_{k \geq n_0}$ satisfies the indicated condition. The condition characterizes the image, because A_n is dense in the set of continuous functions $\mathbb{Z}_p^\times \to \mathbb{Z}_p$.

Now the fact that, p-adically,

$$\lim_{r \to \infty} (p-1)p^r = 0$$

implies that

$$\lim_{r \to \infty} x^{(p-1)p^r} = 1$$

as functions $\mathbb{Z}_p^\times \to \mathbb{Z}_p$. Thus if α is a measure, then

$$\int 1 d\alpha = \lim_r \int x^{(p-1)p^r} d\alpha$$

as indicated. The case of KO is analogous. $\square$

Example 9.8. If c is a p-adic unit, then

$$c^{p-1} \equiv 1 \mod p$$

and so

$$c^{(p-1)p^{k-1}} \equiv 1 \mod p^k.$$

Let $\alpha = \frac{1}{p^k}$; let m and n be integers such that

$$m \equiv n \mod (p-1)p^{k-1};$$

and let

$$h(x) = \alpha(x^m - x^n).$$

Then

$$h(c) \in \mathbb{Z}_p.$$

It follows that if $\{z_n\}_{n \geq n_0}$ is a sequence as in the Proposition, then

$$z_m \equiv z_n \mod p^k$$

if $m \equiv n \mod (p-1)p^{k-1}$. In the case $p = 2$, a slight refinement of this argument shows that

$$\begin{aligned} z_m &\equiv z_n \mod 2 \text{ for all } m, n \\ z_m &\equiv z_n \mod 2^{k+1} \text{ if } k \geq 2 \text{ and } m \equiv n \mod 2^{k-1}. \end{aligned}$$

Example 9.9. Let c be a p-adic unit. By Theorem 10.6, there is a measure μ'_c on $\mathbb{Z}_p^\times/\{\pm 1\}$ with the property that, for k even,

$$\int_{\mathbb{Z}_p^\times/\{\pm 1\}} x^k d\mu'_c = -(1-p^{k-1})(1-c^k)\frac{B_k}{2k}.$$

Moreover, the mean of this measure is

$$\int_{\mathbb{Z}_p^\times/\{\pm 1\}} d\mu'_c = \frac{1}{2p}\log c^{p-1}.$$

It follows that the sequence

$$z_k = -(1-p^{k-1})(1-c^k)\frac{B_k}{2k}$$

for $k \geq 4$ satisfies the conditions of Proposition 9.7. (The congruences of Example 9.8 in this case are known as the *Kummer congruences*) Applying Proposition 9.7 to this sequence, we have the following.

Corollary 9.10. *There is a unique map*

$$g : KO_p \to KO_p$$

such that

$$g_* v^k = -(1-p^{k-1})(1-c^k)\frac{B_k}{2k}v^k$$

for $k \geq 4$. Moreover

$$g_* v^0 = \frac{1}{2p}\log c^{p-1}.$$

10. Bernoulli numbers, Eisenstein series, and the Mazur measure

In this section we assemble some results about Bernoulli numbers and the Mazur measure. Variations on these results are scattered in the literature (see particularly [Ada65, Kat75, Kob77, Ser73]) and are surely known to experts. We include them here because we have not found precisely the results we need by consulting any one source.

10.1. **Bernoulli numbers.** Recall that the Bernoulli numbers are the rational numbers B_k for $k \geq 0$ defined by the formula

$$\frac{x}{e^x - 1} = \sum_{k \geq 0} B_k \frac{x^k}{k!}. \tag{10.1}$$

It is easy to see that $B_0 = 1$ and $B_1 = -1/2$. It is also not difficult to check that $B_k = 0$ for k odd and greater than 1. Indeed this follows from the following result, which relates the Bernoulli numbers to the $\widehat{A}$-genus.

Proposition 10.2.

$$\frac{x}{e^{x/2} - e^{-x/2}} = \exp\left(- \sum_{k \geq 2} \frac{B_k}{k} \frac{x^k}{k!} \right).$$

Proof. We have

$$\begin{aligned} \log\left(\frac{x}{e^{x/2} - e^{-x/2}} \right) &= \log x - \log(e^{x/2}(1 - e^{-x})) \\ &= \log x - \frac{x}{2} - \log(1 - e^{-x}), \end{aligned}$$

and so

$$\begin{aligned} d\log\left(\frac{x}{e^{x/2} - e^{-x/2}} \right) &= \frac{1}{x} - \frac{1}{2} - \frac{e^{-x}}{1 - e^{-x}} \\ &= \frac{1}{x} - \frac{1}{2} - \frac{1}{e^x - 1}. \end{aligned}$$

Comparing with (10.1), we find that

$$d\log\left(\frac{x}{e^{x/2} - e^{-x/2}} \right) = -\sum_{k \geq 2} B_k \frac{x^{k-1}}{k!},$$

and so

$$\frac{x}{e^{x/2} - e^{x/2}} = A \exp\left(- \sum_{k \geq 2} \frac{B_k}{k} \frac{x^k}{k!} \right).$$

Comparison of constant terms shows that $A = 1$. ◻

10.2. **Mazur measure.** In the following, we use the notation

$$\sum_{a \leq i < b}^{*} f(i) \overset{\text{def}}{=} \sum_{\substack{a \leq i < b \\ p \nmid i}} f(i)$$

for a sum over integers not divisible by the given prime p.

Let A denote the set of polynomials $h(x) \in \mathbb{Q}_p[x]$ such that $h(0) = 0$, and such that $h(a) \in \mathbb{Z}_p$ whenever $a \in \mathbb{Z}_p^\times$. Then A is dense in the space $\operatorname{map}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ of continuous functions.

Theorem 10.3. *Fix an element $c \in \mathbb{Z}_p^\times$.*

(a) *There exists a measure μ_c on $\mathbb{Z}_p^\times$,*

$$f(x) \mapsto \int_{\mathbb{Z}_p^\times} f(x)\, d\mu_c(x) \colon \operatorname{map}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \to \mathbb{Z}_p,$$

uniquely characterized by the following property. For $h(x) \in A$, let

$$H_c(x) \overset{def}{=} \int_x^{cx} \frac{h(t)}{t}\, dt.$$

Then

$$\int_{\mathbb{Z}_p^\times} h(x)\, d\mu_c(x) = \lim_{r\to+\infty} \frac{1}{p^r} \sum_{0\le i<p^r}^{*} H_c(i).$$

(b) *For this measure, we have*

$$\int_{\mathbb{Z}_p^\times} x^k\, d\mu_c(x) = -\frac{B_k}{k}(1-p^{k-1})(1-c^k)$$

for $k \ge 1$, and

$$\int_{\mathbb{Z}_p^\times} 1\, d\mu_c(x) = \frac{1}{p}\log(c^{p-1}).$$

The proof will proceed in several steps, given below.

Step 1. Fix a polynomial $h(x) = \sum_{k=1}^K a_k x^k \in A$, so that $H_c(x) = \sum_{k=1}^K \frac{a_k}{k}((cx)^k - x^k)$. Let $-r_0$ be the minimum of the p-adic valuations of the $a_k/k \in \mathbb{Q}_p$, for $1 \le k \le K$. We first show that the p-adic numbers

$$\frac{1}{p^r} \sum_{0\le i<p^r}^{*} H_c(i)$$

are p-adic *integers* for all $r \ge r_0$.

For each integer i such that $0 \le i < p^r$ and $p \nmid i$, there is a unique integer j of the same type such that $ci \equiv j \mod p^r$. Let $m_r(j) \in \mathbb{Z}_p$ be the number making the equation

$$ci = j(1 + m_r(j)p^r)$$

hold. Then for $k \ge 1$,

$$\sum_{0\le i<p^r}^{*} (ci)^k = \sum_{0\le j<p^r}^{*} j^k(1+m_r(j)p^r)^k \equiv \sum_{0\le j<p^r}^{*} j^k + k\, j^k m_r(j)p^r \mod p^{2r}.$$

Thus

$$\sum_{0\le i<p^r}^{*} [(ci)^k - i^k] \equiv \sum_{0\le j<p^r}^{*} k\, j^k m_r(j) p^r \mod p^{2r},$$

hence

$$\frac{1}{p^r}\sum_{0\le i<p^r}^{*} H_c(i) \equiv \sum_{0\le j<p^r}^{*} \sum_{k=1}^{K} m_r(j) a_k j^k \equiv \sum_{0\le j<p^r}^{*} m_r(j)h(j) \mod p^{r-r_0}.$$

By hypothesis, $h(j)$ and $m_r(j)$ are p-adic integers. Thus, we see the expression above is a p-adic integer for all $r \ge r_0$. □

Step 2. We show that if $h(x) = x^k$, $k \ge 1$, then

$$\lim_{r\to+\infty} \frac{1}{p^r}\sum_{0\le i<p^r}^{*} H_c(i) = -\frac{B_k}{k}(1-p^{k-1})(1-c^k).$$

In particular, the limit exists (and by Step 1 must lie in $\mathbb{Z}_p$). This proves the first part of (b).

Define polynomials $F_k(t) \in \mathbb{Q}[t]$ by

$$x\frac{e^{tx}-1}{e^x-1} = \sum_k F_k(t)\frac{x^k}{k!}, \qquad \text{so that} \qquad F_k(t) = \sum_{j=1}^{t} \binom{k}{j} B_{k-j} t^j.$$

(The identity $x\frac{e^{tx}-1}{e^x-1} = \frac{xe^{tx}}{e^x-1} - \frac{x}{e^x-1}$ means that $F_k(t) = B_k(t) - B_k$, where $B_k(t)$ is the Bernoulli polynomial defined by $\frac{xe^{tx}}{e^x-1} = \sum B_k(t)x^k/k!$.)

For $n \ge 0$, let

$$S_k(n) = \sum_{0\le i<n} i^k$$

be the power sum. Then

$$S_k(n) = \frac{1}{k+1} F_{k+1}(n).$$

Note that this is a polynomial in n of form $B_k \cdot n + O(n^2)$.

Define

$$S_k^*(n) = \sum_{0 \le i < n}^* i^k.$$

When $p|n$, we have

$$S_k^*(n) = S_k(n) - p^k S_k(n/p) = \frac{1}{k+1}[F_{k+1}(n) - p^k F_{k+1}(n/p)].$$

This has the form $(1 - p^{k-1})B_k \cdot n + O(n^2)$. It follows that if we let $n = p^r$, and allow $r \to +\infty$, then

$$S_k^*(n)/n \to (1 - p^{k-1})B_k.$$

If $h(x) = x^k$, then $H_c(x) = (c^k - 1)x^k/k$, whence

$$\begin{aligned} \frac{1}{p^r} \sum_{0 \le i < p^r}^* H_c(i) &= \frac{c^k - 1}{k\, p^r} \sum_{0 \le i < p^r}^* i^k \\ &= \frac{c^k - 1}{k} S_k^*(p^r)/p^r \end{aligned}$$

which thus converges to $(1 - p^{k-1})(c^k - 1)B_k/k$ as $r \to +\infty$. □

Step 3. By linearity, Step 2 shows that the limit in (a) converges to an element of $\mathbb{Q}_p$ for all $h \in A$, and by Step 1 this element must lie in $\mathbb{Z}_p$; thus we get a well-defined function $\phi \colon A \to \mathbb{Z}_p$.

It is clear that if $h(x) \in A$ has the property that $h(\mathbb{Z}_p^\times) \subseteq p^r\mathbb{Z}_p$, then $\phi(h) \in p^r\mathbb{Z}_p$. (If h has this property, then $h(x)/p^r \in A$.) Since any continuous function $h \colon \mathbb{Z}_p^\times \to \mathbb{Z}_p$ can be approximated uniformly by a sequence $h_i(x)$ of elements of A, we see that we can define

$$\int_{\mathbb{Z}_p^\times} h(x)\, d\mu_c(x) = \lim_{i \to \infty} \phi(h_i).$$

This constructs the measure of part (a). □

The remaining step is to prove the second formula of part (b). We give the argument in a form which we will also apply in the proof of Proposition 7.15.

To make sense of the statement in part (b), first note that the Taylor series for natural logarithm allows us to define a continuous homomorphism $\log \colon (1 + p\mathbb{Z}_p)^\times \to \mathbb{Z}_p$. This extends in a unique way to a continuous homomorphism $\log \colon \mathbb{Z}_p^\times \to \mathbb{Z}_p$, for instance by the formula

$$\log(a) = \frac{1}{p-1} \log(a^{p-1}).$$

We begin with the following result of von Staudt-Adams.

Lemma 10.4. *If p is odd, and $k \equiv 0 \mod p-1$, or if $p = 2$ and k is even, then*

$$\frac{B_k}{k}(1 - p^{k-1}) \equiv \left(1 - \frac{1}{p}\right)\frac{1}{k} \mod \mathbb{Z}_p.$$

□

Proof. This is easily deduced from Theorem 2.5 of [Ada65]. □

For convenience, we let

$$N(k) = (p-1)p^k.$$

Proposition 10.5. *Let $b_k \in \mathbb{Q}_p$ be a sequence satisfying*

$$b_k \equiv -\frac{B_k}{k} \mod \mathbb{Z}_p,$$

and let c be a p-adic unit. Then

$$\lim_{k\to\infty} (1 - p^{N(k)-1})(1 - c^{N(k)})b_k = \frac{1}{p} \log c^{p-1}.$$

Proof. By Lemma 10.4 we have

$$(1 - p^{N(k)-1})b_{N(k)} \equiv -\frac{1}{N(k)} \left(1 - \frac{1}{p}\right) \mod \mathbb{Z}_p,$$

and since

$$1 - c^{N(k)} \equiv 0 \mod p^k,$$

we have

$$(1 - c^{N(k)})(1 - p^{N(k)-1})b_{N(k)} \equiv -\frac{1 - c^{N(k)}}{N(k)} \left(1 - \frac{1}{p}\right) \mod p^k.$$

Now

$$\lim_{k\to\infty} \frac{1 - c^{N(k)}}{N(k)} = -\frac{1}{p-1} \log c^{p-1},$$

so

$$\lim_{k\to\infty} (1 - p^{N(k)-1})(1 - c^{N(k)})b_{N(k)} = \frac{p-1}{p}\frac{1}{p-1} \log c^{p-1} = \frac{1}{p} \log c^{p-1},$$

as required. □

This completes the proof of Theorem 10.3. □

10.3. **Half measures.**

Theorem 10.6. *Let μ_c be the measure constructed in (10.3). There exists a measure μ_c' on $\mathbb{Z}_p^\times/\{\pm 1\}$, characterized by the property that*

$$\int_{\mathbb{Z}_p^\times/\{\pm 1\}} h(x)\, d\mu_c'(x) = \frac{1}{2} \int_{\mathbb{Z}_p} h(x)\, d\mu_c(x),$$

for all $h \in \operatorname{map}(\mathbb{Z}_p/\{\pm 1\}, \mathbb{Z}_p)$.

Proof. The existence of μ_c' is clear when p is odd; only the case $p = 2$ requires explanation.

Let A' be the set of polynomials $h(x) \in \mathbb{Q}_2[x]$ such that $h(x) = h(-x)$ and $h(0) = 0$, and such that $h(a) \in \mathbb{Z}_2$ whenever $a \in \mathbb{Z}_2^\times$. Then A' is dense in the space $\operatorname{map}(\mathbb{Z}_2^\times/\{\pm 1\}, \mathbb{Z}_2)$. Fix a polynomial $h(x) = \sum_{k=2}^K a_k x^k \in A'$. Let $-r_0$ be the minimum of the 2-adic valuations of the $a_k/k \in \mathbb{Q}_2$, for $2 \le k \le K$. Consider $H_c(t) = \int_x^{cx} h(t)/t\, dt$. I claim that

$$\frac{1}{2^{r+1}} \sum_{0\le i<2^r}^{*} H_c(i) \in \mathbb{Z}_2$$

for all sufficiently large r. Given this, it is clear that the desired measure exists, by the arguments of the previous section.

For each integer i such that $0 \le i < 2^r$ and $2 \nmid i$, there is a unique integer j of the same type such that $ci \equiv \pm j \mod 2^{r+1}$. Let $q_r(j) \in \{\pm 1\}$ and $m_r(j) \in \mathbb{Z}_2$ be the unique elements satisfying the equation

$$ci = q_r(j) j(1 + m_r(j)2^{r+1}).$$

Then for even integers $k \ge 2$,

$$\sum_{0\le i<2^r}^{*} (ci)^k = \sum_{0\le j<2^r}^{*} q_r(j)^k j^k (1 + m_r(j)2^{r+1})^k \equiv \sum_{0\le j<2^r}^{*} j^k + kj^k m_r(j)2^{r+1} \mod 2^{2r+2}.$$

Thus

$$\sum_{0\le i<2^r}^{*}[(ci)^k - i^k] \equiv \sum_{0\le j<2^r}^{*} k\, j^k m_r(j) 2^{r+1} \mod 2^{2r+2},$$

and thus

$$\frac{1}{2^{r+1}}\sum_{0\le i<2^r}^{*} H_c(i) \equiv \sum_{0\le j<2^r}^{*}\sum_{k=2}^{K} m_r(j) a_k j^k \equiv \sum_{0\le j<2^r}^{*} m_r(j)h(j) \mod 2^{r-r_0+1}.$$

As before, $m_r(j), h(j) \in \mathbb{Z}_2$, so that this expression is a 2-adic integer when $r \ge r_0 - 1$. □

Note that this result and Proposition 10.5 imply the following.

Corollary 10.7. *Let $b_k \in \mathbb{Q}_p$ be a sequence satisfying*

$$b_k \equiv -\frac{B_k}{2k} \mod \mathbb{Z}_p.$$

Then

$$\lim (1 - p^{N(k)-1})(1 - c^{N(k)}) b_k = \frac{1}{2p}\log c^{p-1},$$

and this quantity is a p-adic integer. □

10.4. **Eisenstein series.** Recall that the *normalized Eisenstein series* are the power series G_k given by

$$G_k = -\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n) q^n \tag{10.8}$$

if k is even and $G_k = 0$ if k is odd, where

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

For $k > 1$, G_{2k} is the q-expansion of a modular form. These Eisenstein series are related to the sigma orientation of because of the following: note that the left-hand-side below is a form of the Weierstrass sigma function (see for example [AHS04]).

Proposition 10.9.

$$\frac{x}{e^{x/2} - e^{-x/2}}\prod_{n\ge 1}\frac{(1-q^n)^2}{(1-q^n e^x)(1-q^n e^{-x})} = \exp\left(\sum_{k\ge 2} 2G_k \frac{x^k}{k!}\right).$$

Proof. Let's write $u = e^x$ for convenience. We have

$$\begin{aligned}\log\left(\prod_{n\ge1}\frac{(1-q^n)^2}{(1-q^n u)(1-q^n u^{-1})}\right) &= 2\sum_{n\ge1}\log(1-q^n) - \sum_{n\ge1}\log(1-q^n u) - \sum_{n\ge1}\log(1-q^n u^{-1})\\ &= -2\sum_{n\ge1}\sum_{d\ge1}\frac{q^{nd}}{d} + \sum_{n\ge1}\sum_{d\ge1}\frac{q^{nd}}{d}(u^d + u^{-d})\\ &= -2\sum_{n\ge1}\sum_{d\ge1}\frac{q^{nd}}{d} + \sum_{n\ge1}\sum_{d\ge1}\frac{q^{nd}}{d}\sum_{k\ge0}\frac{1}{k!}d^k(x^k + (-1)^k x^k).\end{aligned}$$

In this expression, the coefficient of x^k is 0 if k is 0 or odd, and, if k is even,

$$\frac{2}{k!}\sum_{d\ge1} d^{k-1}\sum_{n\ge1} q^{nd},$$

which is the same as

$$\frac{2}{k!}\sum_{n\ge1}\sigma_{k-1}(n) q^n.$$

Together with the equation

$$\frac{x}{e^{x/2}-e^{-x/2}} = \exp\left(-\sum_{k\geq 2} \frac{B_k}{k}\frac{x^k}{k!}\right)$$

which we prove in Proposition 10.2, we have

$$\frac{x}{e^{x/2}-e^{-x/2}} \prod_{n\geq 1} \frac{(1-q^n)^2}{(1-q^n e^x)(1-q^n e^{-x})} = \exp\left(\sum_{k\geq 2} 2G_k \frac{x^k}{k!}\right),$$

as required. ∎

10.5. **Kummer congruences for Eisenstein series.** Let p be a prime. Then we write

$$G_k^*(q) = G_k(q) - p^{k-1}G_k(q^p)$$

and

$$\sigma_{k-1}^*(n) = \sum_{d|n,(p,d)=1} d^{k-1}.$$

It is easy to check that

$$G_k^*(q) = -(1-p^{k-1})\frac{B_k}{2k} + \sum_{n\geq 1} \sigma_{k-1}^*(n) q^n.$$

Once again it is convenient to let $N(r) = (p-1)p^r$.

Proposition 10.10. *For each prime p and for each p-adic unit c, there is a unique measure ν_c on $\mathbb{Z}_p^\times/\{\pm 1\}$, taking values in $MF_{p,*}$, with moments*

$$\int_{\mathbb{Z}_p^\times/\{\pm1\}} x^k d\nu_c = (1-c^k)G_k^*$$

for k even and greater than three. Moreover this measure has mean

$$\lim_{r\to\infty}(1-c^{N(r)})G_{N(r)}^* = \frac{1}{2p}\log c^{p-1}.$$

Proof. Fix a prime p. Let

$$h(z) = \sum a_k z^k \in \mathbb{Q}_p[z]$$

be a test polynomial for the generalized Kummer congruences (Definition 9.6), i.e. for every p-adic unit c,

$$\sum a_k c^k \in \mathbb{Z}_p.$$

Fix a p-adic unit c. We must show that

$$\sum a_k(1-c^k)G_k^* \in \mathbb{Z}_p[\![q]\!].$$

The constant term (coefficient of q^0) in $(1-c^k)G_k^*$ is

$$-(1-c^k)(1-p^{k-1})\frac{B_k}{2k},$$

and so the generalized Kummer congruences for this term is equivalent to the existence of the Mazur measure as explained in Example 9.9. For $n \geq 1$ the coefficient of q^n in $(1-c^k)G_k^*$ is

$$(1-c^k)\sigma_{k-1}^*(n) = (1-c^k)\sum_{d|n,(p,d)=1} d^{k-1} = \frac{1}{d}(1-c^k)\sum_{d|n,(p,d)=1} d^k.$$

Note that the d in the sum are p-adic units.

$$\sum_k a_k(1-c^k)\sigma_{k-1}^*(n) = \sum_{d|n,(p,d)=1} d^{-1}\left(\sum_k a_k 1 - \sum_k a_k (cd)^k\right)$$

The description of h means that each of the terms $\sum_k a_k$ and $\sum_k a_k(cd)^k$ is an element of $\mathbb{Z}_p$, and so the whole expression is as well.

The second part follows from Corollary 10.7. Explicitly, if c is a p-adic unit, then

$$c^{p-1} \equiv 1 \mod p$$

and so

$$c^{N(r)} \equiv 1 \mod p^r,$$

i.e.

$$|1 - c^{N(r)}|_p = \frac{1}{p^r}.$$

Since $|\sigma^*_{k-1}(n)|_p \leq 1$, we have

$$|(1 - c^{N(r)})\sigma^*_{N(r)-1}(n)| \leq \frac{1}{p^r}.$$

Thus

$$\lim_{r\to\infty} (1 - c^{N(r)})G^*_{N(r)} = \lim_{r\to\infty} -(1 - c^{N(r)})(1 - p^{k-1})\frac{B_k}{2k} = \frac{1}{2p} \log c^{p-1},$$

by Proposition 10.5. □

11. $K(n)$ localizations of tmf

Proposition 11.1. $\pi_{2k} tmf \otimes \mathbb{Q} \cong MF_k \otimes \mathbb{Q}$ *and* $\pi_{2k}(tmf^\wedge_p) \otimes \mathbb{Q} \cong MF_k \otimes \mathbb{Q}_p$.

Lemma 11.2. $tmf^\wedge_p$ *is torsion in odd degrees.*

Lemma 11.3.

$$tmf^\wedge_p \cong (L_2 tmf^\wedge_p)\langle 0, \ldots, \infty\rangle$$

We need the following facts about $K(n)$-localizations of tmf. If g is a power series

$$g = \sum a_n q^n,$$

let $g|V$ and $g|U$ be the power series

$$g|U = \sum a_n q^{pn}$$
$$g|V = \sum a_{pn} q^n.$$

Thus if g is a p-adic modular form, then V and U are the Verschiebung and Atkin operators. If $g_k \in MF_k$ is a modular form of weight k, then the Hecke operator $T(p)$ is given by

$$g_k|T(p)(q) = g_k|U + p^{k-1} g_k|V,$$

and

$$g^*_k = g_k - p^{k-1} g_k|V$$

is a p-adic modular form of weight k. [Ser73]

The logarithm ℓ_1 for $L_{K(1)} tmf$ is a map

$$gl_1 tmf \to L_{K(1)} gl_1 tmf \xrightarrow{\ell_1} L_{K(1)} tmf.$$

Proposition 11.4. i) *The natural map*

$$\pi_{2k} tmf \to MF_k$$

induces a map

$$\pi_{2k} L_{K(1)} tmf \to MF_{p,k}$$

from the homotopy of the $K(1)$-localization of tmf to the ring of p-adic modular forms.

ii) *The θ-algebra structure of $L_{K(1)} tmf$ is such that the diagram*

$$\begin{array}{ccc} \pi_{2k} L_{K(1)} tmf & \longrightarrow & MF_{p,k} \\ \psi \downarrow & & \downarrow g \mapsto p^k g|V \\ \pi_{2k} L_{K(1)} tmf & \longrightarrow & MF_{p,k} \end{array} \tag{11.5}$$

iii) *There is an operation $U : L_{K(1)}tmf \to L_{K(1)}tmf$ making the diagram*

$$\begin{tikzcd}
\pi_{2k}L_{K(1)}tmf \arrow[r, "\pi_{2k}U"] \arrow[d] & \pi_{2k}L_{K(1)}tmf \arrow[d] \\
MF_{p,k} \arrow[r, "U"] & MF_{p,k}
\end{tikzcd}$$

commute.

iv) $[KO_p, L_{K(1)}tmf_p^\wedge]$ *is torsion free. Indeed, the natural map*

$$[bstring, L_{K(1)}tmf_p^\wedge] \cong [KO_p, L_{K(1)}tmf_p^\wedge] \to \prod_{k \geq 4} MF_{p,k}$$

is an isomorphism onto the set of sequences of p-adic modular forms b_k (with b_k of weight k) which satisfy the generalized Kummer congruences. (Definition 9.6) If $\{b_k\}$ is such a sequence, corresponding to a map $f : KO_p \to L_{K(1)}tmf_p^\wedge$, then

$$\pi_0 f = \lim_r b_{(p-1)p^r}$$

v) $[KO_p, L_{K(1)}L_{K(2)}tmf_p^\wedge]$ *is torsion free, and the natural map*

$$[KO_p, L_{K(1)}L_{K(2)}tmf_p^\wedge] \cong [bstring, L_{K(1)}L_{K(2)}tmf_p^\wedge] \to$$
$$[bstring, L_{K(1)}L_{K(2)}tmf_p^\wedge \otimes \mathbb{Q}] \xrightarrow{s} \mathbf{D}\left(L_{K(1)}L_{K(2)}tmf_p^\wedge\right)$$

is injective.

Now consider the diagram

$$\begin{tikzcd}
L_{K(1)}gl_1tmf \arrow[r] \arrow[d, "\ell_1"', "\approx"] & L_{K(1)}L_{K(2)}gl_1tmf \arrow[d, "\approx"', "L_{K(1)}\ell_2"] \\
L_{K(1)}tmf \arrow[r, "b"] & L_{K(1)}L_{K(2)}tmf,
\end{tikzcd}$$

where the top horizontal arrow is $L_{K(1)}$ applied to $K(2)$-localization, and the bottom horizontal arrow b is defined so that the diagram commutes.

Proposition 11.6. i) *The diagram*

$$\begin{tikzcd}
\pi_{2k}gl_1tmf \arrow[r] \arrow[d, "\ell_1"'] & MF_k \arrow[d, "g_k \mapsto g_k^*"] \\
\pi_{2k}L_{K(1)}tmf \arrow[r] & MF_{p,k}
\end{tikzcd}$$

commutes.

ii) *The diagram*

$$\begin{tikzcd}
\pi_{2k}L_{K(1)}tmf \arrow[rr, "\pi_{2k}b"] \arrow[dr, "\pi_{2k}(1-U)"'] & & \pi_{2k}L_{K(1)}L_{K(2)}tmf \\
& \pi_{2k}L_{K(1)}tmf, \arrow[ur] &
\end{tikzcd}$$

where the bottom right arrow is $L_{K(1)}$ applied to $K(2)$ localization.

Proof. Substituting the formula (11.5) for ψ in the equation (4.7) for ℓ_1 shows that, if g is a modular form of weight k representing an element $1 + g \in \pi_{2k}gl_1tmf$, then

$$\ell_1 g = (1 - p^{k-1}V)g.$$

On the other hand, Proposition 4.8 implies that

$$\ell_2 g = (1 - T(p) + R)g = (1 - T(p) + p^{n-1})g.$$

Note that $UVg = g$. Thus

$$\begin{aligned}(1-U)\ell_1 g &= (1-U)(1-p^{n-1}V)g \\ &= (1-U-p^{k-1}V+p^{k-1}UV)g \\ &= (1-T(p)+p^{k-1})g \\ &= \ell_2 g,\end{aligned}$$

as required. □

12. String orientations of tmf: statement of main results

To state a result about orientations of tmf, we write MF_k for the group of integral modular forms of weight k and level 1. Then $MF_* \otimes \mathbb{Q}$ is isomorphic to the ring of rational modular forms of level one. A modular form $f \in MF_k \otimes \mathbb{Q}$ has a q-expansion

$$f(q) \in \sum_n f_n q^n.$$

If f is a modular form and p is a prime, then we write $f|T(p)$ for the modular form given by applying the Hecke operator $T(p)$ to f. Recall that if f has weight k, then

$$f|T(p)(q) = \sum_n f_n q^{pn} + p^{k-1} \sum_n f_{pn} q^n.$$

We write B_k for the k-the Bernoulli number, defined by

$$\frac{x}{e^x - 1} = \sum_{k \geq 0} B_k \frac{x^k}{k!}.$$

Theorem 12.1. *$\mathbf{C}(gl_1 tmf)$ is the set of sequences $(g_k \in MF_k \otimes \mathbb{Q})_{k \geq 4}$ satisfying the following three conditions.*

i) $g_k(q) \equiv -\frac{B_k}{2k} \mod \mathbb{Z}[\![q]\!]$.

ii) *Given a prime p, write $g_k^*(q) = g_k(q) - p^{k-1} g_k(q^p)$. For each prime p and each $c \in \mathbb{Z}_p^\times$, the sequence $\{(1-c^k) g_k^*(q))\}_{k \geq 4}$ satisfies the* generalized Kummer congruences *(Definition 9.6).*

iii) *For all primes p, $g_k|T(p) = (1+p^{k-1}) g_k$.*

Remark 12.2. Note that condition iii) implies that each g_k is a multiple of the Eisenstein series G_k, since this is the eigenfunction of $T(p)$ with the indicated eigenvalue [Ser70].

For $k \geq 4$ let $G_k \in MF_k \otimes \mathbb{Q}$ be the Eisenstein series, normalized as

$$G_k = -\frac{B_k}{2k} + \sum_{n=1} \sigma_{k-1}(n) q^n,$$

where

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

Theorem 12.3. *The sequence $\{G_k\}_{k \geq 4}$ satisfies the conditions of Theorem 12.1, and so $\pi_0 E_\infty(MString, tmf)$ is nonempty. The E_∞ orientations with characteristic series $\{G_k\}_{k \geq 4}$ refine the sigma orientation of* [AHS04].

13. First reductions

In this section we reduce the study of orientations of tmf to a problem about $K(1)$-local spectra.

Proposition 13.1. *Applying $\pi_0 \mathbf{A}(gl_1(-))$ to the homotopy pull-back square*

$$\begin{array}{ccc} tmf & \longrightarrow & \prod_p tmf_p^\wedge \\ \big\downarrow & & \big\downarrow \\ tmf \otimes \mathbb{Q} & \longrightarrow & \left(\prod_p tmf_p^\wedge\right) \otimes \mathbb{Q} \end{array} \tag{13.2}$$

yields a Cartesian square

$$\begin{array}{ccc} \pi_0 \mathbf{A}(gl_1 tmf) & \longrightarrow & \prod_p \pi_0 \mathbf{A}\left(gl_1 tmf_p^\wedge\right) \\ \big\downarrow & & \big\downarrow \\ \mathbf{D}(gl_1 tmf) & \longrightarrow & \mathbf{D}\left(\prod_p gl_1 tmf_p^\wedge\right). \end{array}$$

In particular $\mathbf{C}(gl_1 tmf) \subseteq \mathbf{D}(gl_1 tmf) \cong \mathbf{D}(tmf)$ is the set of sequences $(g_k \in MF_k \otimes \mathbb{Q})$ such that, for each prime p, the image of this sequence in $MF_* \otimes \mathbb{Q}_p$ lies in $\mathbf{C}\left(gl_1 tmf_p^\wedge\right)$.

Proof. By Proposition 5.19, applying $\mathbf{A}(gl_1(-))$ to the diagram (13.2) yields a homotopy pull-back diagram. Let

$$Y = \prod_p gl_1 tmf_p^\wedge$$

$$X = \left(\prod_p gl_1 tmf_p^\wedge\right) \otimes \mathbb{Q}.$$

Since X is rational, and $(gl_1 S) \otimes \mathbb{Q} \approx *$, we have a weak equivalence

$$\mathbf{A}(X) \approx \mathrm{map}(bstring, X). \tag{13.3}$$

From this we get a Mayer-Vietoris sequence

$$[\Sigma bstring, X] \to \pi_0 \mathbf{A}(gl_1 tmf) \to \pi_0 \mathbf{A}(gl_1 tmf \otimes \mathbb{Q}) \oplus \pi_0 \mathbf{A}(Y) \twoheadrightarrow [bstring, X].$$

But $[\Sigma bstring, X] \cong [bstring, \Sigma^{-1} X] \cong \mathbf{D}\left(\Sigma^{-1} X\right) = 0$, because for all p, $tmf_p^\wedge$ is torsion in odd degrees (Lemma 11.2). ☐

Proposition 13.4. *The natural map*

$$\mathbf{A}\left(gl_1 tmf_p^\wedge\right) \to \mathbf{A}\left(L_{K(1)\vee K(2)} gl_1 tmf_p^\wedge\right) \tag{13.5}$$

is a weak equivalence, and so there is a homotopy pull-back square

$$\begin{array}{ccc} \mathbf{A}\left(gl_1 tmf_p^\wedge\right) & \longrightarrow & \mathbf{A}\left(L_{K(2)} gl_1 tmf_p^\wedge\right) \\ \big\downarrow & & \big\downarrow \\ \mathbf{A}\left(L_{K(1)} gl_1 tmf_p^\wedge\right) & \longrightarrow & \mathbf{A}\left(L_{K(1)} L_{K(2)} gl_1 tmf_p^\wedge\right). \end{array}$$

Proof. Since (Lemma 11.3)

$$tmf_p^\wedge \cong (L_2 tmf_p^\wedge)\langle 0, \dots, \infty\rangle$$

we have

$$gl_1 tmf_p^\wedge \cong gl_1 L_2 tmf_p^\wedge.$$

Theorem 4.11 implies that

$$\pi_q \,\mathrm{fib}(gl_1 L_2 tmf_p^\wedge \to L_2 gl_1 L_2 tmf_p^\wedge)$$

is torsion, and zero for $q > 2$, and so Proposition 5.19 implies that (13.5) is a weak equivalence. ☐

For the next result, we recall the following.

Lemma 13.6. *$K(2) \wedge bstring$ and $K(2) \wedge bu\langle 6\rangle$ are contractible.* ☐

Proposition 13.7. *For all X under $gl_1 S$, $\mathbf{A}\left(L_{K(2)}X\right) \approx *$, and the image of $\pi_0 \mathbf{A}\left(L_{K(2)}X\right)$ in $\mathbf{D}\left(L_{K(2)}X\right)$ is the zero sequence.*

Proof. Since $K(2) \wedge bstring \approx *$, $gl_1 S \to gl_1 S/string$ is a $K(2)$-equivalence. This proves the first part.

In particular, up to homotopy there is a unique map g making the diagram

$$\begin{tikzcd} string \arrow[r, "j"] & gl_1 S \arrow[r, "\approx_{K(2)}"] \arrow[d] & gl_1 S/string \arrow[r] \arrow[dl, "g"] & bstring \\ & L_{K(2)}X, & & \end{tikzcd}$$

in which the row is a cofibration, commute. The second statement in the proposition is that the characteristic map of g is the zero sequence.

To see this, suppose that k is an even number greater than or equal to 4, and suppose that $x_k \in \pi_{2k} gl_1 S/string$ maps to a non-zero element y_k of $\pi_{2k} bstring$. Let $\lambda_k \in \mathbb{Q}$ be defined by the formula

$$r_* y_k = \lambda_k v^k$$

in $\pi_{2k} bu \otimes \mathbb{Q}$. Then $b_k = b_k(g)$ is determined

$$g_* x_k = \lambda_k b_k(g) \in \pi_{2k}(L_{K(2)}X) \otimes \mathbb{Q}.$$

Since g factors through $L_{K(2)} gl_1 S/string$, to prove the second part it is enough to show x_k maps to a torsion element of $\pi_{2k} L_{K(2)} gl_1 S/string$.

There are a variety of ways to show this. For example, Friedlander's proof of the stable Adams Conjecture [Fri80] implies that, if c is a generator of $\mathbb{Z}_p^{\times}$, then there is a map F_c making the diagram

$$\begin{tikzcd} bu\langle 6\rangle \arrow[r, equal] \arrow[dd, "F_c"'] & bu\langle 6\rangle \arrow[r] \arrow[d, "\psi^c - 1"] & * \arrow[dd] \\ & bu\langle 6\rangle \arrow[d] & \\ gl_1 S/string \arrow[r] & bstring \arrow[r] & bgl_1 S \end{tikzcd} \tag{13.8}$$

in which the rows are cofiber sequences, commute. We can take x_k to be the image of $v^k \in \pi_{2k} bu\langle 6\rangle$ under F_c. Then x_k maps to zero in $L_{K(2)} gl_1 S/string$, since $K(2) \wedge bu\langle 6\rangle \approx *$. ☐

14. Orientations of $L_{K(1)} tmf$

Propositions 13.4 and 13.7 together imply that we have a fibration

$$\mathbf{A}\left(gl_1 tmf_p^{\wedge}\right) \to \mathbf{A}\left(L_{K(1)} gl_1 tmf_p^{\wedge}\right) \to \mathbf{A}\left(L_{K(1)} L_{K(2)} gl_1 tmf_p^{\wedge}\right). \tag{14.1}$$

In this section we analyze $\mathbf{A}\left(L_{K(1)} gl_1 tmf_p^{\wedge}\right)$. The analysis is similar to our analysis of $\mathbf{A}\left(KO_p\right)$ in §7. In §15 we analyze the map in the sequence (14.1).

Note that Theorem 4.11 and Proposition 5.19 together imply that

$$\mathbf{A}\left(gl_1 L_{K(1)} tmf_p^{\wedge}\right) \approx \mathbf{A}\left(L_{K(1)} gl_1 L_{K(1)} tmf_p^{\wedge}\right).$$

At the same time, we have the equivalences

$$L_{K(1)} gl_1 L_{K(1)} tmf_p^{\wedge} \xrightarrow[\approx]{\ell_1} L_{K(1)} tmf_p^{\wedge} \xleftarrow[\approx]{\ell_1} L_{K(1)} gl_1 tmf_p^{\wedge}.$$

Putting these together, we find that

$$\mathbf{A}\left(L_{K(1)} gl_1 tmf_p^{\wedge}\right) \approx \mathbf{A}\left(gl_1 L_{K(1)} tmf_p^{\wedge}\right)$$

and so we our description of $\mathbf{A}\left(L_{K(1)}gl_1 tmf_p^{\wedge}\right)$ yields a description of the E_∞ string orientation fo $L_{K(1)}tmf_p^{\wedge}$:

Lemma 14.2. *$\mathbf{A}\left(L_{K(1)}gl_1 tmf_p^{\wedge}\right)$ is homotopy equivalent to the space of E_∞ maps*

$$MString \to L_{K(1)}tmf_p^{\wedge}.$$

$\square$

The analysis proceeds much as our analysis of $E_\infty(MString, KO_p)$. To begin with, we have the following.

Lemma 14.3. *Let $X = L_{K(1)}gl_1 tmf_p^{\wedge}$ or $L_{K(1)}L_{K(2)}gl_1 tmf_p^{\wedge}$. The maps*

$$\pi_0\mathbf{A}\left(X\right) \to \mathbf{B}\left(X\right) \to \mathbf{C}\left(X\right)$$

are bijections.

Proof. One possibility is that all three sets are empty. If not, then recall from Proposition 5.19 that we have surjections

$$\pi_0\mathbf{A}\left(X\right) \twoheadrightarrow \mathbf{B}\left(X\right) \twoheadrightarrow \mathbf{C}\left(X\right),$$

with "kernel" the torsion subgroup of $[bstring, X] = [KO_p, X]$. But if

$$X = L_{K(1)}gl_1 tmf_p^{\wedge} \approx L_{K(1)}tmf_p^{\wedge} \text{ or } L_{K(1)}L_{K(2)}gl_1 tmf_p^{\wedge} \approx L_{K(1)}L_{K(2)}tmf_p^{\wedge},$$

then $[KO_p, X]$ is torsion-free (Proposition 11.4). $\square$

Remark 14.4. The main difference from the argument for KO_p is that I have not assumed that

$$[\Sigma^{-1}KO_p, L_{K(1)}tmf_p^{\wedge}] = 0.$$

To study $\pi_0\mathbf{A}\left(L_{K(1)}gl_1 tmf_p^{\wedge}\right) \cong \mathbf{B}\left(L_{K(1)}gl_1 tmf_p^{\wedge}\right)$, consider the diagram

$$\begin{tikzcd}
L_{K(1)}S \arrow[r, "\rho(c)"] & KO_p \arrow[r, "1-\psi^c"] \arrow[d, "\Phi B_c"', "\approx"] & KO_p \arrow[d, equal] & \\
L_{K(1)}gl_1 S \arrow[u, "\ell_1", "\approx"'] \arrow[r] \arrow[d] \arrow[dr, "L_{K(1)}gl_1\iota"'] & L_{K(1)}gl_1 S/string \arrow[r] \arrow[d, dashed, "\alpha"'] & KO_p \arrow[r] \arrow[d, dashed, "\beta"'] \arrow[dr, "\mathbf{m}_{gl_1 tmf_p^{\wedge}}"] & L_{K(1)}bgl_1 S \arrow[d] \\
\Sigma^{-1}L_{K(1)}gl_1 tmf_p^{\wedge} \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "\ell_1"'] & L_{K(1)}gl_1 tmf_p^{\wedge} \arrow[r] \arrow[d, "\ell_1"', "\approx"] & L_{K(1)}gl_1 tmf_p^{\wedge} \otimes \mathbb{Q} \arrow[r, "r"'] \arrow[d, "\ell_1"', "\approx"] & L_{K(1)}gl_1 tmf_p^{\wedge} \otimes \mathbb{Q}/\mathbb{Z} \arrow[d, "\ell_1"', "\approx"] \\
\Sigma^{-1}L_{K(1)}tmf_p^{\wedge} \otimes \mathbb{Q}/\mathbb{Z} \arrow[r] & L_{K(1)}tmf_p^{\wedge} \arrow[r] & L_{K(1)}tmf_p^{\wedge} \otimes \mathbb{Q} \arrow[r] & L_{K(1)}tmf_p^{\wedge} \otimes \mathbb{Q}/\mathbb{Z},
\end{tikzcd} \tag{14.5}$$

which is modified from (7.8) in two ways: first, by replacing KO with tmf as the target, and second, by displaying the map $L_{K(1)}gl_1 S \to L_{K(1)}S$ as in (7.13). We recall that in Proposition 7.15 we proved that

$$\rho(c)^{-1} = \frac{1}{2p}\log(c^{p-1}) \in \pi_0 KO_p.$$

For convenience let

$$N(r) = (p-1)p^r.$$

Proposition 14.6. *The characteristic map*

$$\pi_0\mathbf{A}\left(L_{K(1)}gl_1 tmf_p^{\wedge}\right) \to \mathbf{D}\left(L_{K(1)}gl_1 tmf_p^{\wedge}\right)$$

is an isomorphism to the set of sequences of p-adic modular forms $\{b_k \in MF_{p,2k}\}_{k\geq 4}$ such that, for all p-adic units c,

(1) *the sequence $\{(1-c^k)b_k^*\}_{k\geq 4}$ satisfies the generalized Kummer congruences (9.6).*
(2) $\lim_{r\to\infty}(1-c^{N(r)})b_{N(r)}^* = \rho(c)^{-1}$.

Proof. The argument is much the same as it was in the case of KO_p in §7. Proposition 11.4 says that to give a map

$$\alpha : L_{K(1)} gl_1 S/string \to L_{K(1)} gl_1 tmf_p^{\wedge} \tag{14.7}$$

is equivalent to giving an sequence

$$\{t_k(\alpha) \stackrel{\text{def}}{=} \pi_{2k}(\ell_1 \alpha \Phi B_c)\}_{k \geq 4} \in \prod_{kgeq4} \pi_{2k} L_{K(1)} tmf_p^{\wedge}$$

of p-adic modular forms satisfying the generalized Kummer congruences.

On the other hand, $b(\alpha)$ is the sequence

$$\{b_k(\alpha) = \pi_{2k}\beta\}_{k\geq 4} \in \prod_{k\geq 4} \pi_{2k} gl_1 tmf_p^{\wedge} \otimes \mathbb{Q}$$

defined using the map β in the diagram (14.5). To compare these it is convenient first to consider $b(\ell_1\alpha)$.

Proposition 11.6 shows that

$$b_k(\ell_1\alpha) = b_k^*(\alpha).$$

Inspection of the diagram (14.5) shows that

$$t_k(\alpha) = (1 - c^k) b_k^*(\alpha).$$

Thus to give a pair of maps α and β making the middle square in (14.5) commute is equivalent to giving a sequence $\{b_k(\alpha)\}_{k\geq 4} \in \pi_* tmf_p^{\wedge} \otimes \mathbb{Q}$ such that the sequence $\{(1-c^k)b_k^*(\alpha)\}_{k\geq 4}$ satisfies the generalized Kummer congruences.

Such α, β make the whole diagram (14.5) commute, and so correspond to an element of $\mathbf{B}\left(L_{K(1)} gl_1 tmf_p^{\wedge}\right)$ (and, incidentally, an E_∞ orientation $MString \to L_{K(1)} tmf_p^{\wedge}$), if and only if

$$\ell_1 \alpha \Phi B_c \rho(c) = 1$$

By Proposition 11.4, this is the condition that

$$\lim_{r\to\infty} (1 - c^{N(r)}) b_{N(r)}^*(\alpha) = \rho(c)^{-1}.$$

□

Remark 14.8. Once we know that $\mathbf{A}\left(L_{K(1)} gl_1 tmf_p^{\wedge}\right)$ is non-empty, we can proceed as in §7. The condition involving $\rho(c)$ can be replaced with the condition

$$b_k \equiv G_k \mod \mathbb{Z}.$$

See the proof of Theorem 12.1, below.

15. Orientations of tmf

Proposition 15.1. *$\mathbf{C}\left(gl_1 tmf_p^{\wedge}\right)$ is the set of sequences $(g_k) \in \prod_{k\geq 4} MF_{p,2k}$ such that*

1. *the sequence (g_k^*) lies in $\mathbf{C}\left(L_{K(1)} tmf\right)$ (that is, it satisfies the conditions of Proposition 14.6); and*
2. *$g_k^*|(1-U) = 0$.*

Proof. Propositions 13.4 and 13.7 imply that the sequence

$$\pi_0 \mathbf{A}\left(gl_1 tmf_p^{\wedge}\right) \xrightarrow{\gamma} \pi_0 \mathbf{A}\left(L_{K(1)} gl_1 tmf_p^{\wedge}\right) \xrightarrow{\delta} \pi_0 \mathbf{A}\left(L_{K(1)} L_{K(2)} gl_1 tmf_p^{\wedge}\right)$$

is exact in the middle, in the sense that $\operatorname{im}\gamma = \delta^{-1}(*)$, where $*$ is the element in the image of $\pi_0\mathbf{A}\left(L_{K(2)} gl_1 tmf\right)$. Using Proposition 14.3, we may replace $\pi_0\mathbf{A}\left(\,-\,\right)$ with $\mathbf{C}\left(\,-\,\right)$ for the $K(1)$-localizations.

Proposition 13.7 shows that under

$$\pi_0 \mathbf{A}\left(L_{K(1)} L_{K(2)} gl_1 tmf_p^{\wedge}\right) \to \mathbf{C}\left(L_{K(1)} L_{K(2)} gl_1 tmf_p^{\wedge}\right),$$

$*$ maps to the zero sequence. Proposition 11.6 shows that the diagram

$$\begin{array}{ccc} \mathbf{C}\left(L_{K(1)}gl_1 tmf_p^\wedge\right) & \longrightarrow & \mathbf{C}\left(L_{K(1)}L_{K(2)}gl_1 tmf_p^\wedge\right) \\ \ell_1 \downarrow & & \downarrow L_{K(2)}\ell_2 \\ \mathbf{C}\left(L_{K(1)}tmf_p^\wedge\right) & \xrightarrow{1-U} & \mathbf{C}\left(L_{K(1)}L_{K(2)}tmf_p^\wedge\right) \end{array}$$

commutes, giving the result. ☐

Recall [AHS01] that the the sigma orientation is an orientation of elliptic spectra which refines the Witten genus, whose Hirzebruch series is

$$\frac{x}{e^{x/2}-e^{-x/2}}\prod\frac{(1-q^n)^2}{(1-q^ne^x)(1-q^ne^{-x})}=\exp\left(\sum_{k\geq 2}2G_k\frac{x^k}{k!}\right).$$

In Proposition 10.9 we show that

$$\frac{x}{\exp_\sigma(x)}=\exp\left(2\sum_k G_k\frac{x^k}{k!}\right),$$

where G_k is the normalized Eisenstein series

$$G_k=-\frac{B_k}{2k}+\sum_{n\geq 1}\sigma_{k-1}(n)q^n.$$

Here

$$\sigma_{k-1}(n)=\sum_{d|n}d^{k-1},$$

and if we also define

$$\sigma^*_{k-1}(n)=\sum_{d|n,(p,d)=1}d^{k-1}$$

then it is easy to check that

$$G^*_k=-(1-p^{k-1})\frac{B_k}{2k}+\sum_{n\geq 1}\sigma^*_{k-1}(n)q^n. \tag{15.2}$$

Proposition 15.3. *The sequence $\{G_k\}_{k\geq 4}$ satisfies the conditions of Proposition 15.1, and so is the characteristic map of an E_∞ orientation $MString\to tmf_p^\wedge$.*

Proof. The p-adic conditions of Proposition 15.1 are proved as Proposition 10.10. The remaining condition is that $G^*_k|(1-U)=0$, which follows easily from (15.2). ☐

We can now prove that $\pi_0E_\infty(MString, tmf)$ contains an orientation which refines the sigma orientation.

Proof of Theorem 12.3. According to Proposition 13.1, $\mathbf{C}(gl_1tmf)$ is the set of sequences $b_k\in MF_k\otimes\mathbb{Q}$ of modular forms such that, for all p, $\{b_k\}\in\mathbf{C}\left(gl_1tmf_p^\wedge\right)\}$. We have just shown that $\{G_k\}$ is such a sequence. ☐

Corollary 15.4. *The Miller invariant of $gl_1S\to gl_1tmf$ satisfies*

$$(\mathbf{m}_{gl_1tmf})_*v^k\equiv G_k\equiv-\frac{B_k}{2k}\mod\mathbb{Z}.$$

☐

Proof of Theorem 12.1. Proposition 13.1 and Proposition 15.1 together identify $\mathbf{C}(gl_1tmf)$ with the set of sequences $\{g_k\in MF_k\otimes\mathbb{Q}\}$ of modular forms such that, for all primes p and units $c\in\mathbb{Z}_p^\times$,

(1) $g^*_k|(1-U)=0$;
(2) the sequence $\{(1-c^k)g^*_k\}_{k\geq 4}$ satisfies the generalized Kummer congruences; and
(3) $\lim_{r\to\infty}(1-c^{p^r}(p-1))g^*_{p^r(p-1)}=\rho(c)^{-1}$.

Noting that

$$g_k^*|(1-U) = g_k|(1-p^{k-1}V)|(1-U) = g_k|(1-T(p)+p^{k-1}),$$

the condition $g_k^*|(1-U) = 0$ gives the condition involving $T(p)$ in the statement of the Theorem. The condition involving Kummer congruences in the statement of the Theorem is identical to the one here. For the last condition, note that on the one hand, the characteristic map b of an orientation u satisfies

$$b_k(u) \equiv -\frac{B_k}{2k} \mod \mathbb{Z}$$

by Corollary 15.4. On the other hand, by Corollary 10.7, if

$$g_k \equiv -\frac{B_k}{2k} \mod \mathbb{Z},$$

then

$$\lim_{r\to\infty}(1-c^{N(r)})g^*_{N(r)} = \rho(c)^{-1},$$

as required. □

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