Categorical tori and their representations

A report on work in progress

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Crossed modules and categorical groups following Noohi

(Strict) categorical groups are (strict) monoidal groupoids G_1 $s \downarrow \downarrow t$ G_0 with invertible objects (w.r.t. •).

A crossed module (G, H, ψ) encodes a strict categorical group

$$\begin{array}{c}
G \ltimes H \\
pr_1 & \downarrow & pr_1 \cdot \psi \\
G
\end{array}$$

group multiplication gives • and $(g\psi(k), h) \circ (g, k) = (g, hk).$

Crossed modules

consist of groups G and H, a right action of G on H by automorphisms and a homomorphism $\psi: H \longrightarrow G$ with

$$\psi(h^g) = g^{-1}\psi(h)g$$

$$k^{\psi(h)} = h^{-1}kh.$$

The crossed module of the categorical group ${\mathcal G}$ above is

$$G = G_0$$

$$H = ker(s)$$

$$h^g = g^{-1} \bullet h \bullet g$$

$$\psi = t.$$

Example: the crossed module of a categorical torus

Two ingredients: A lattice Λ^{\vee} and a bilinear form J on Λ^{\vee} . From this, we form the crossed module

$$\Lambda^{\!\!\vee} imes U(1) \xrightarrow{\psi} \mathfrak{t} := \Lambda^{\!\!\vee} \otimes_{\mathbb{Z}} \mathbb{R}$$
 $(m, z) \longmapsto m,$

where the action of $x \in \mathfrak{t}$ on $\Lambda^{\vee} \times U(1)$ is given by

$$(m,z)^{\times} = (m,z \cdot e(J(m,x))).$$

Here and throughout the talk, we use the notation

$$e(r) := e^{2\pi i r}, \qquad r \in \mathbb{R}.$$

Categorical tori

The categorical torus \mathcal{T} is the strict monoidal category with

objects: t,

arrows:
$$x \xrightarrow{z} x + m, \quad x \in \mathfrak{t}, m \in \Lambda^{\vee}, z \in U(1),$$

composition: the obvious one,

multiplication: addition on objects and on arrows

$$(x \xrightarrow{z} x + m) \bullet (y \xrightarrow{w} y + n) = (x + y \xrightarrow{zw \cdot e(J(m, y))} x + y + m + n).$$

Classification

Schommer-Pries, Wagemann-Wockel, Carey-Johnson-Murray-Stevenson-Wang

Up to equivalence, the categorical torus ${\mathcal T}$ only depends on the even symmetric bilinear form

I(m,n) = J(m,n) + J(n,m).

More precisely,

 $-I \in Bil_{ev}(\Lambda^{\vee},\mathbb{Z})^{S_2} = H^4(BT;\mathbb{Z}) \cong H^3_{gp}(T;U(1))$

classifies the equivalence class of the extension

 $pt/\!\!/ U(1) \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}.$

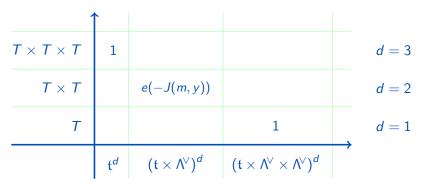
Examples:

- T_{max} ⊂ G maximal torus of a simple and simply connected compact Lie group, Λ^V coroot lattice, I_{bas} basic bilinear form,
 (A = 1) an another Nieuroparties
- 2. (Λ_{Leech}, I) or another Niemeyer lattice.

Aussie-rules Lie group cohomology

 $H^3_{gp}(T; U(1)) = \check{H}^3(BT_{\bullet}; \underline{U(1)})$

and -1 corresponds to the Čech-simplicial 3-cocycle



where the non-trivial entry is short for

 $((x,m),(y,n)) \longmapsto e(-J(m,y)).$

Autoequivalences of the category of coherent sheaves

 $\widehat{T} = Hom(T, U(1))$

 $T_{\mathbb{C}} = \operatorname{spec} \mathbb{C}[\widehat{T}]$

 $\mathcal{C}ohT_{\mathbb{C}} \simeq \mathbb{C}[\widehat{T}] - \mathit{mod}^{\mathit{fin}}$

$$1Aut(\mathcal{C}ohT_{\mathbb{C}}) \simeq \left(Bimod_{\mathbb{C}[\widehat{T}]}^{fin}
ight)^{ imes}$$
 [Deligne].

Inside $1Aut(Coh(T_{\mathbb{C}}))$, we have the full subcategory of direct image functors f_* of variety automorphisms f. This categorical group belongs to the crossed module

$$\mathbb{C}[\widehat{T}]^{\times} \xrightarrow{1} Aut_{var}(T_{\mathbb{C}}),$$

where f acts on $\mathbb{C}[\widehat{T}]^{\times}$ by precomposition, $\varphi \mapsto \varphi \circ f$.

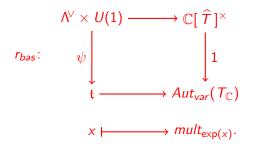
The basic representation of a categorical torus

The basic representation of \mathcal{T} is the strict monoidal functor

 $\varrho_{bas}: \mathcal{T} \longrightarrow 1Aut(\mathcal{C}oh(\mathcal{T}_{\mathbb{C}})).$

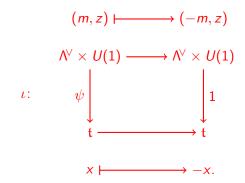
induced by the map of crossed modules

 $(m, z) \longmapsto z \cdot e^{2\pi i J(m, -)}$



The involution ι

The involution ι of T, sending t to t^{-1} lifts to an involution of T, given by the map of crossed modules



This gives rise to an action of the group $\{\pm 1\}$ by (strict monoidal) functors on the category \mathcal{T} .

Extraspecial categorical 2-groups

The fixed points of ι on T form the elementary abelian 2-group

 $T^{\{\pm 1\}} = T[2] \cong \Lambda^{\vee} / 2\Lambda^{\vee}.$

The categorical fixed points (or equivariant objects) of ι on ${\cal T}$ form an extension

$$pt/\!\!/ U(1) \longrightarrow \mathcal{T}^{\{\pm 1\}} \longrightarrow \widetilde{T[2]}$$

of the extraspecial 2-group $\widetilde{T[2]}$ with Arf invariant

$$\phi(m) = \frac{1}{2}I(m,m) \mod 2\Lambda^{\vee}.$$

Example: In the example of the Leech lattice, T[2] is the subgroup of the Monster that is usually denoted 2^{1+24} .

1Automorphisms of the basic representation

Let $\mathcal{T}_{\mathbb{C}} \rtimes \{\pm 1\}$ be the categorical group of the crossed module

$$\begin{array}{c} \Lambda^{\!\!\vee}\times\mathbb{C}^{\times} & \longrightarrow & \mathfrak{t}_{\mathbb{C}} \rtimes \{\pm 1\} \\ (m,z) & \longmapsto & (m,1), \end{array}$$

where -1 acts on everything by ι .

Extend the basic representation to

 $\varrho_{bas}: \mathcal{T}_{\mathbb{C}} \rtimes \{\pm 1\} \longrightarrow 1Aut(Coh(T_{\mathbb{C}})),$

by setting $r_{bas}(-1) := \iota$. So, $\varrho_{bas}(-1) := \iota_*$.

Theorem: The 1automorphisms of this ρ_{bas} form the extraspecial categorical 2-group $\mathcal{T}_{\mathbb{C}}^{\{\pm 1\}}$.

Normalizers

Let

$$\varrho: H \longrightarrow G = GL(V)$$

be a representation of a group H on some vector space. Then

$$Aut(\varrho) = C(\varrho) = \{g \in G \mid c_g \circ \varrho = \varrho\}$$

is the centralizer of (the image of) ϱ in G. Here c_g is conjugation by g.

Definition [Dror Farjoun, Segev]: The *injective normalizer* of ρ is the subgroup of $Aut(H) \times G$ defined as

 $N(\varrho) = \{(f,g) \mid c_g \circ \varrho = \varrho \circ f\}.$

If ϱ is injective, this is the normalizer of its image.

Towards the refined Monster? (In progress)

Theorem: The 1automorphisms of \mathcal{T} form an extension

$$pt/\!\!/\Lambda \longrightarrow 1Aut(\mathcal{T}) \longrightarrow O(\Lambda^{\!\vee}, I).$$

Here $O(\Lambda^{\vee}, I)$ is the group of linear isometries of (Λ^{\vee}, I) .

Example: the Conway group

 $O(\Lambda_{Leech}^{\vee}, I) = Co_0.$

In spirit, the subgroup of the Monster, known as

$$2^{1+24}.Co_1 = \widetilde{T[2]} \rtimes (Co_0 / \{\pm id\})$$

wants to parametrize the isomorphism classes of some categorical variant of normalizer of ρ_{bas} : $\mathcal{T}_{\mathbb{C}} \rtimes \{\pm 1\} \longrightarrow 1Aut(Coh(\mathcal{T}_{\mathbb{C}}))$.