2 Matrices and linear equations

Chapter contents

2.1 Arithmetic with matrices

A *matrix* is a two-dimensional rectangular array of (to start with, real) numbers, called *entries*:

$$\begin{bmatrix} 1 & 2 & 3 \\ \frac{1}{2} & 1.3 & 0 \end{bmatrix}.$$

This is a 2×3 matrix, which indicates that it has 2 rows and 3 columns.

A general matrix is often denoted

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

If $m = n$, we say that the matrix is *square*, and we call the entries $a_{11}, a_{22}, \dots, a_{nn}$ the *(main) diagonal*.

A matrix of size $1 \times n$ is a *row matrix*. A matrix of size $n \times 1$ is a *column matrix*.

A *zero matrix* is a matrix all of whose entries are zero. We will denote it $\mathbf{0}$ to avoid confusing it with the number 0.

Scalar multiplication

Let $A = [a_{ij}]$ be a matrix and λ be a scalar. We define a new matrix $\lambda A = [\lambda a_{ij}]$.

Example 2.1. Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 2 & -7 \end{bmatrix}, \qquad \text{then} \qquad -2A =$$

Addition of matrices

Given two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size, their *sum* is the matrix $A + B = [a_{ij} + b_{ij}]$.

Example 2.2.

$$\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \qquad\qquad \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} =$$

Some properties of scalar multiplication and addition

$$\begin{aligned}
(\lambda + \mu)C &= \lambda C + \mu C \\
\lambda(B + C) &= \lambda B + \lambda C \\
\lambda(\mu C) &= (\lambda\mu)C \\
0C &= \mathbf{0} \\
A + B &= B + A \\
A + (B + C) &= (A + B) + C \\
A - A &= \mathbf{0} \\
A + \mathbf{0} &= A
\end{aligned}$$

Matrix multiplication

Suppose $A = [a_{ij}]$ has size $m \times n$ and $B = [b_{ij}]$ has size $n \times p$. Then we define an $m \times p$ matrix $AB = [c_{ij}]$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Example 2.3.

$$\begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -7 \end{bmatrix} = \qquad\qquad \begin{bmatrix} 4 \\ -7 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 0 \\ 0 & 1 \end{bmatrix} =$$

Example 2.4.

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \qquad\qquad \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} =$$

So, in general, $AB \neq BA$. We say that A and B *commute* if $AB = BA$. Of course this requires that A and B be square of the same size (but this is not sufficient, as we just saw).

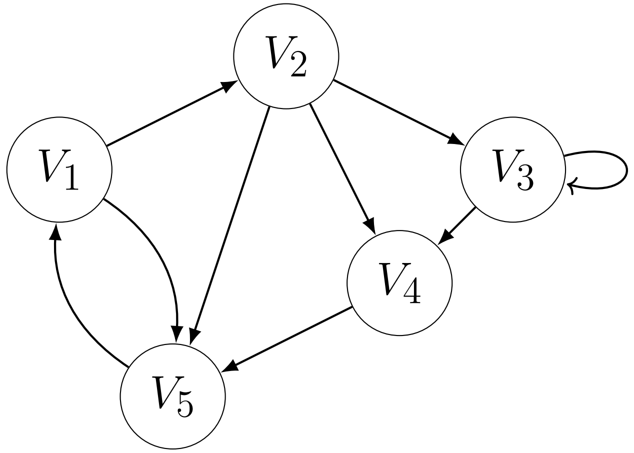
The *identity matrix* I_n is the $n \times n$ matrix with 1s on the diagonal and 0s everywhere else.

Some properties of matrix multiplication

$$\begin{aligned} A(B+C) &= AB + AC \\ (A+B)C &= AC + BC \\ A(BC) &= (AB)C \\ AI &= IA = A \\ A\mathbf{0} &= \mathbf{0}A = \mathbf{0} \end{aligned}$$

Application: Adjacency matrix of a graph

A *graph* is a finite set of points called *vertices* together with a finite set of arrows called *edges* that connect certain vertices.



$A =$

The connectivity information of a graph can be recorded in its *adjacency matrix* A; the (i, j) entry of the matrix (row i, column j) is equal to the number of edges that go from V_i to V_j.

Theorem 2.5. *The* (i, j) *entry of* A^n *is equal to the number of paths of length* n *that start at* V_i *and end at* V_j*.*

In our example, what is the number of paths of length 3 from V_2 to V_5?

Properties of matrix multiplication, continued

Matrix multiplication can be used to extract the columns of a matrix:

$$A\mathbf{e}_1 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} =$$

This gives us another way to look at the multiplication of a matrix by a vector:

$$A\mathbf{b} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} =$$

This is called a *linear combination* of the columns of A, and is a type of expression that plays a crucial role in linear algebra.

2.2 Using matrices to solve linear systems

An equation is *algebraic* if it involves only constants, variables, and algebraic operations (addition, subtraction, multiplication). An algebraic equation is *linear* if each variable occurs to the power one only. A *linear system* is a set of linear equations.

The equation

$$2x + y = 5$$

describes a line in the plane $\mathbf{R}^2$. It has *solution set*

$$\{(x,y) \in \mathbf{R}^2 :\ x = \qquad\qquad y = \qquad\qquad\qquad\qquad\qquad\qquad \},$$

where t is called a *free parameter*.

We will prefer to write this in the form

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \qquad\quad \\ \\ \end{bmatrix} = \begin{bmatrix} \ \\ \\ \end{bmatrix} + t \begin{bmatrix} \quad\ \\ \\ \end{bmatrix}, \quad t \in \mathbf{R} \right\}.$$

Example 2.6. A system can have more equations and more variables:

$$\begin{cases} -x_1 + 2x_2 + x_3 - 3x_4 = 1 \\ \ \ x_1 \qquad\quad\ \ + x_3 - \ \ x_4 = 3 \\ 3x_1 + \ \ x_2 \qquad\quad + 3x_4 = -1 \end{cases}$$

This can be written in *matrix form*

$$A\mathbf{x} = \mathbf{b},$$

The *augmented matrix* of the system puts A and $\mathbf{b}$ together:

Think of solving a system as a game. We make moves that take us to a system that has the same solutions but is easier to solve.

There are three types of allowed moves, called *elementary row operations* on the augmented matrix:

(a) Multiply a row by a nonzero constant: $R \leftarrow \lambda R$, $\lambda \neq 0$.

(b) Swap two rows: $R \leftrightarrow S$.

(c) Add a multiple of one row to another: $R \leftarrow R + \mu S$.

What is a winning position in this game? It is a reduced row echelon form, defined as follows.

For a nonzero row of a matrix, the *leading entry* is the leftmost nonzero entry in the row.

A matrix is in *reduced row echelon form (RREF)* if

- Lower leading entries appear to the right of higher leading entries.
- All zero rows are grouped at the bottom of the matrix.
- All leading entries are equal to 1.
- Any column that contains a leading entry has all the other entries 0.

Example 2.7 (Some matrices in various states of reduceness).

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \qquad \text{are}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{are}$$

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{is}$$

Example 2.8. For the augmented matrix of the system in Example 2.6 we have

$$\begin{bmatrix} -1 & 2 & 1 & -3 & 1 \\ 1 & 0 & 1 & -1 & 3 \\ 3 & 1 & 0 & 3 & -1 \end{bmatrix}$$

According to the rules, we have won the game. Have we really? Let's consider the system corresponding to the last matrix:

$$\begin{cases} x_1 \qquad\qquad\qquad\quad + \; x_4 = 0 \\ \qquad x_2 \qquad\qquad\qquad\quad = -1 \\ \qquad\qquad\quad x_3 - 2x_4 = 3. \end{cases}$$

This is trivial to solve!

Take   as a free parameter, call it t, then the solution set is

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \qquad\qquad\qquad t \in \mathbf{R} \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \qquad\qquad\qquad\qquad t \in \mathbf{R} \right\}.$$

You should check that this is a solution of the original system.

Gaussian elimination is an algorithm for reducing any matrix to a matrix in row-echelon form:

Step 1: Find the row with the leftmost leading entry in the entire matrix. (If necessary) interchange rows to make this row the first row of the matrix.

Step 2: Make all entries below this leading entry into 0s by adding/subtracting suitable multiples of the first row to the lower rows.

Step 3: Ignore the first row of the matrix, and start over from **Step 1**.

One matrix can give rise to **many different** row echelon forms (depending on choices made during Gaussian elimination).

For instance, although this is not really necessary, it is common to scale the leading entries to 1 as they are found, since this (often) simplifies the subsequent calculations.

Gauss–Jordan elimination is an extension that adds two further steps to Gaussian elimination, so that the end result is in reduced row echelon form:

Step 4: For each leading entry, make all the entries above it 0 by adding/subtracting a suitable multiple of this row to the rows above.

Step 5: Multiply each nonzero row by the inverse of its leading entry, so that in the end all leading entries are 1.

The most efficient way to apply **Step 4** is to start at the bottom of the matrix and work toward the top.

There is a **unique** reduced row echelon form for any given matrix.

When applied to the augmented matrix of a linear system, Gauss–Jordan elimination allows us to solve the system and find out whether

- the system has a unique solution, or
- the system has infinitely many solutions, and parametrise them, or
- the system has no solutions (is *inconsistent*).

We have seen an instance of an infinite set of solutions in Example 2.8, with the result described on Page 14.

Elementary row operations are actually matrix multiplications!

Elementary matrices

An $n \times n$ matrix is said to be *elementary* if it can be obtained by applying a single elementary row operation (Page 11) to the identity matrix I_n.

Example 2.9. The matrix

$$E = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

is elementary, resulting from

Example 2.10. What matrix do we obtain from

$$M = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

after

(a) multiplying it on the left by E?

(b) applying the row operation $R_2 \leftarrow R_2 - 2R_1$?

This holds for all elementary row operations, and all matrix sizes.

A matrix A is *invertible* if there exists a matrix B such that

$$AB = BA = I_n. \tag{1}$$

Proposition 2.11. *If a matrix B as in Equation (1) exists, then it is unique.*

If it exists, the unique matrix B is called the *inverse* of A and is denoted A^{-1}.

If A and B are $n \times n$ matrices, then $AB = I_n$ is equivalent to A and B being inverses of each other. (We will prove this later.)

For an illustration of why invertible matrices are good, consider:

Proposition 2.12. *Let M and N be $m \times n$ matrices. Let S_M be the linear system with augmented matrix M; let S_N be the linear system with augmented matrix N. Suppose $N = AM$, where A is an invertible $m \times m$ matrix.*

Then the solution set of S_M is equal to the solution set of S_N.

A row operation ρ is *invertible* if there exists a row operation π that undoes its effect.

Example 2.13. A non-example is the (non-elementary) row operation $\rho: \ R \leftarrow 0R$, because this results in the zero row, and there is no way to recover the lost information.

Proposition 2.14. *Any elementary row operation is invertible.*

Proof. We consider each type of elementary row operation in turn:

(a) If ρ is $R \leftarrow \lambda R$ with $\lambda \neq 0$, we can take π

(b) If ρ is $R \leftrightarrow S$, we can take π

(c) If ρ is $R \leftarrow R + \lambda S$, we can take π

$\square$

Proposition 2.15. *Any elementary matrix is invertible.*

Example 2.16. Find the relevant sequence of matrix multiplications:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & 2 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_1} \qquad\qquad \xrightarrow{R_2 \leftarrow (-R_2)} \qquad\qquad \xrightarrow{R_3 \leftarrow R_3 + 2R_2}$$

Upshot: if a matrix N is the result of applying elementary row operations $\rho_1, \rho_2, \ldots, \rho_{n-1}, \rho_n$ (in this order) to a matrix M, then

$$N = E_n E_{n-1} \ldots E_2 E_1 M,$$

where E_i is the elementary matrix corresponding to ρ_i.

Combining this with Proposition 2.12 we deduce the correctness of Gaussian elimination:

Theorem 2.17. *The system associated with a matrix M has the same solution set as the system associated to any row echelon form of M.*

Rank, invertibility, and solvability

The *rank* of a matrix in row echelon form is defined to be the number of leading entries (which is the same as the number of nonzero rows).

The *rank* of an arbitrary matrix A is the rank of any REF matrix obtained from A via Gaussian elimination.

The rank of the matrix from Example 2.8 is 3, while in Example 2.21 we will see a matrix of rank 2.

Theorem 2.18. *Let A be an $n \times n$ matrix. The following are equivalent (TFAE):*

(a) A is invertible.

(b) A has rank n. (We also say it has full rank.)

(c) The RREF of A is I_n.

(d) The homogeneous system $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.

(e) Given any $n \times 1$ matrix $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. (If $\mathbf{b} \neq \mathbf{0}$, we call the system inhomogeneous.)

Example 2.19. Consider the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

(a) Find a row echelon form of $[A \mid \mathbf{b}]$.

(b) Find the rank of the matrix A.

(c) Find all the solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

(d) For which values of a, b, and c does the system $A\mathbf{x} = \mathbf{b}$ have infinitely many solutions, a unique solution, or no solutions?

In general, given an inhomogeneous system $A\mathbf{x} = \mathbf{b}$, if

$$\operatorname{rank} A < \operatorname{rank}[A \mid \mathbf{b}],$$

then the system has no solutions (it is inconsistent).

Note also, in part (c), that

$$\operatorname{rank} A + \#\text{free parameters} = \#\text{unknowns} = \#\text{columns}.$$

Inverting a matrix

Given an $n \times n$ matrix A, we want to find the inverse of A, if it exists.

The proof of Theorem 2.18 gives us an algorithm for finding the inverse of a matrix A, if the latter is invertible:

1. Apply Gauss–Jordan elimination on $[A \mid I]$ to get a matrix $[C \mid D]$.
2. If $C \neq I$, then A is not invertible.
3. If $C = I$, then A is invertible and $A^{-1} = D$.

Example 2.20. Decide whether the following matrix is invertible. If yes, find its inverse.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 2 & 2 & 1 \end{bmatrix}.$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Example 2.21 (A non-invertible matrix)**.**

$$[A \mid I] = \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 3 & -1 & 4 & 0 & 0 & 1 \end{array}\right]$$

Example 2.22. Using the result of Example 2.20, solve

$$\begin{cases} x \phantom{{}-y} + z = 0 \\ 2x - y + 3z = 0 \\ 2x + 2y + z = 0 \end{cases}$$

2.3 Determinants

Motivation: detecting invertibility

We know how to find the inverse of a square matrix (if this inverse exists) by using Gauss–Jordan elimination. In the case of 2×2 matrices, we can give a simple direct formula for the inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

which should be interpreted as saying:

So the number $ad - bc$ detects whether the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible. We call it the *determinant* of the 2×2 matrix A and denote it $\det A$.

We want to explore this notion beyond the special case of 2×2 matrices.

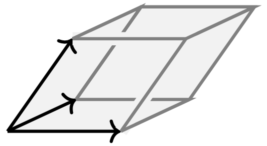
Better motivation: volumes

The rows of an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

are

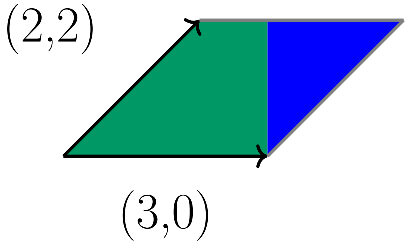
Goal: Define and calculate the *oriented volume* of the *parallelotope* (higher dimensional parallelogram) given by the rows of A.

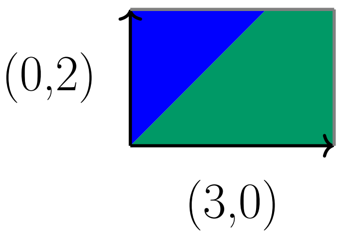


We will call this the *determinant* of A, denoted $\det(A)$ (or simply $|A|$, although I personally avoid this notation).

Adding a multiple of one row to another

$A = \begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

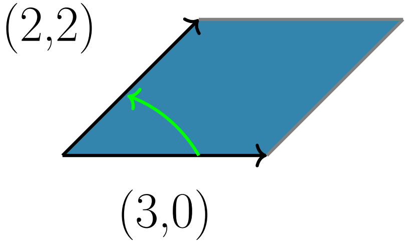


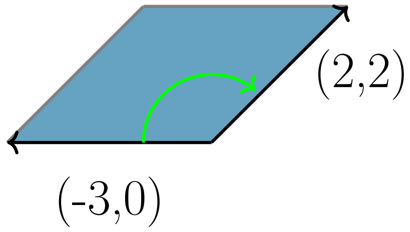


Same area.

Multiplying a row by -1

$A = \begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

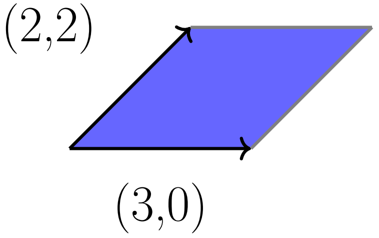


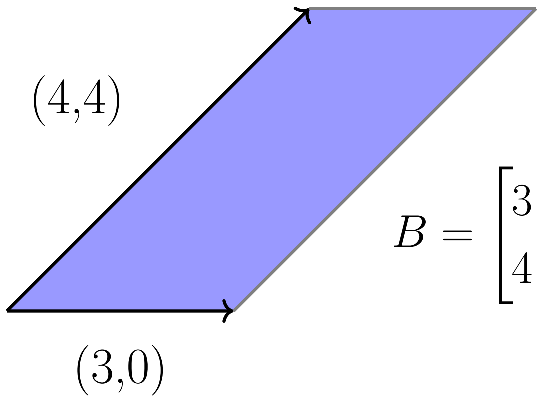


Same area, opposite orientation.

Multiplying one row by a (positive) scalar

$A = \begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix}$



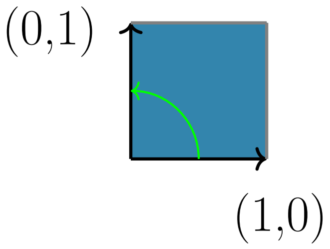


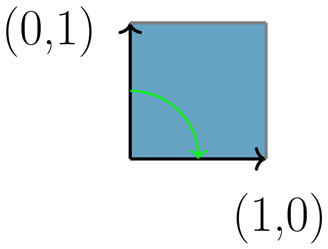
$B = \begin{bmatrix} 3 & 0 \\ 4 & 4 \end{bmatrix}$

Doubling one row doubles the area.

Swapping two rows

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$





Same area, opposite orientation.

Write $M_{n\times n}$ for the set of all $n \times n$ matrices.

In light of the geometric exploration of areas in the previous slides, let's describe the *determinant* as a function $\det : M_{n\times n} \longrightarrow \mathbf{R}$ that behaves according to the following four axioms:

(D1) Adding one row to another row does not change the determinant.

(D2) Multiplying a row by a scalar α multiplies the determinant by α.

(D3) Swapping two rows multiplies the determinant by -1.

(D4) The determinant of the identity matrix is $\det(I_n) = 1$.

Let us assume (for the moment) that such a function exists, and try to find out more about it.

Proposition 2.23. *Let A be an $n \times n$ matrix.*

(a) Adding a multiple of one row to another row does not change the determinant.

(b) If A has at least one zero row, then $\det A = 0$.

(c) If A has two equal rows, then $\det A = 0$.

(d) If α is a scalar and A is an $n \times n$ matrix, then $\det(\alpha A) = \alpha^n \det(A)$.

Proposition 2.24. *The determinant of a diagonal matrix A is equal to the product of the diagonal entries of A.*

Proposition 2.25. *The determinant of an upper triangular matrix A is equal to the product of the diagonal entries of A.*

Theorem 2.26. *Gaussian elimination computes the determinant of arbitrary* $n \times n$ *matrices.*

Example 2.27. Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 6 \\ 7 & 8 & 8 \end{bmatrix}.$$

More properties of the determinant

We mention these without proof:

- Uniqueness: axioms (D1)–(D4) uniquely determine the determinant function, which is why we speak of “the” determinant of A instead of “a” determinant of A.
- $\det(A) = \det\left(A^T\right)$ for any $n \times n$ matrix A.
- Multiplicativity: $\det(AB) = \det(A)\,\det(B)$ for any $n \times n$ matrices A and B.
- Suppose a square matrix A has a *block decomposition*

  $$A = \left[\begin{array}{c|c} B & * \\ \hline 0 & C \end{array}\right],$$

  where B and C are square matrices (not necessarily of the same size). Then

  $$\det(A) = \det(B)\,\det(C).$$

Caution: $\det(A + B) \neq \det(A) + \det(B)$ in general.

The multiplicativity of the determinant function allows us to address our first motivation for studying determinants: detecting matrix invertibility.

Corollary 2.28. *A square matrix A is invertible if and only if* $\det A \neq 0$. *Moreover, if* $\det(A) \neq 0$ *then* $\det(A^{-1}) = \det(A)^{-1}$.

Determinants via cofactor expansion

Consider an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

The *(i,j)-submatrix* of A, denoted A_{ij}, is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i-th row and the j-th column:

$$A_{ij} = \begin{bmatrix} a_{11} & \dots & a_{1(j-1)} & a_{1j} & a_{1(j+1)} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & & \\ a_{(i-1)1} & \dots & a_{(i-1)(j-1)} & a_{(i-1)j} & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ a_{i1} & \dots & a_{i(j-1)} & a_{ij} & a_{i(j+1)} & \dots & a_{in} \\ a_{(i+1)1} & \dots & a_{(i+1)(j-1)} & a_{(i+1)j} & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & & \\ a_{n1} & \dots & a_{n(j-1)} & a_{nj} & a_{n(j+1)} & \dots & a_{nn} \end{bmatrix}.$$

The *(i,j)-minor* of A is $\det A_{ij}$, and the *(i,j)-cofactor* of A is $C_{ij} = (-1)^{i+j} \det A_{ij}$.

Example 2.29. For the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

- $A_{23} =$
- the $(2, 3)$-minor is
- the $(2, 3)$-cofactor is $C_{23} =$

Cofactor (or Laplace) expansion takes a square matrix and returns a number, in one of the following ways:

(a) along row i:

$$\sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(b) along column j:

$$\sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Theorem 2.30. *Cofactor expansion on a square matrix A computes the determinant of A.*

Cofactor expansion has a recursive aspect, whereby the determinant of a larger matrix is written in terms of determinants of smaller matrices (which, in turn, are expressed in terms of determinants of even smaller matrices). This makes statements about cofactor expansion particularly well-suited for proofs by induction.

Pro-tip: remembering the sign of the cofactor

The $(1,1)$ cofactor always has sign $+$. Starting from there, imagine walking to the square you want using either horizontal or vertical steps. The appropriate sign changes at each step.

We can visualise this “chessboard” arrangement with the following matrix:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

So, for example, C_{13} is assigned $+$ but C_{32} is assigned $-$.

Example 2.31. Use cofactor expansion to compute the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

The cofactor method is particularly useful for matrices with many zeros.

Example 2.32. Calculate $\begin{vmatrix} 1 & -2 & 0 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -4 & 2 & 4 \end{vmatrix}$ using cofactor expansion.