6 Introduction to multivariable calculus

Chapter contents

6.1 Functions of two variables

A *real-valued function of two variables* is a function $f: D \longrightarrow \mathbf{R}$, where $D \subseteq \mathbf{R}^2$.

Example 6.1. The volume of a cylinder of radius r and height h is

The *graph of $f: D \subseteq \mathbf{R}^2 \longrightarrow \mathbf{R}$* is the surface

$$\{(x, y, z) \in \mathbf{R}^3 \mid z = f(x, y)\}.$$

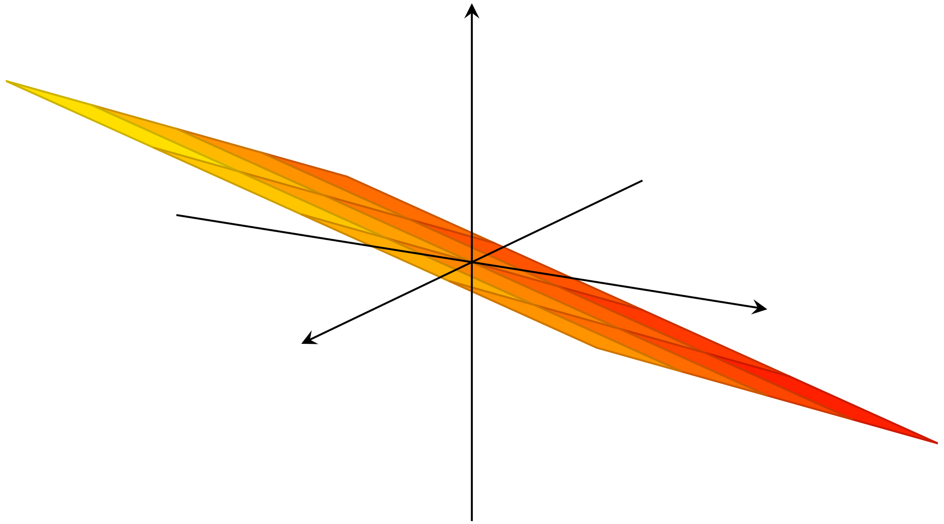
At a point $(x, y) \in D$, $z = f(x, y)$ gives the height of the corresponding point on the surface.

The *level curves of $f: D \subseteq \mathbf{R}^2 \longrightarrow \mathbf{R}$* are the subsets of D of the form

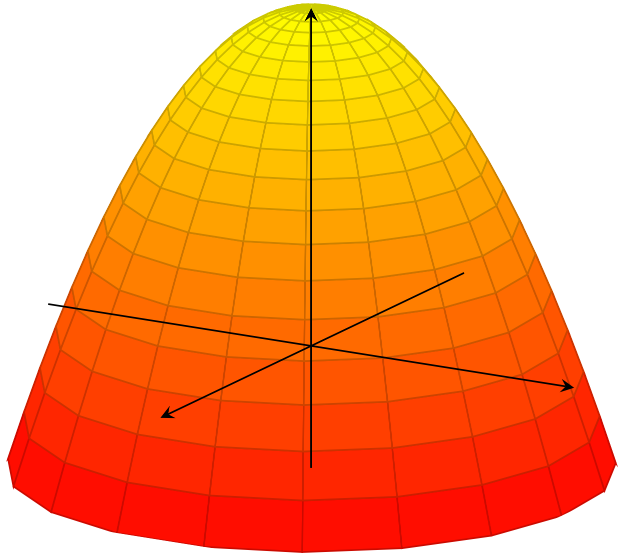
$$\{(x, y) \in D \mid f(x, y) = C\}$$

for different values of the constant C.

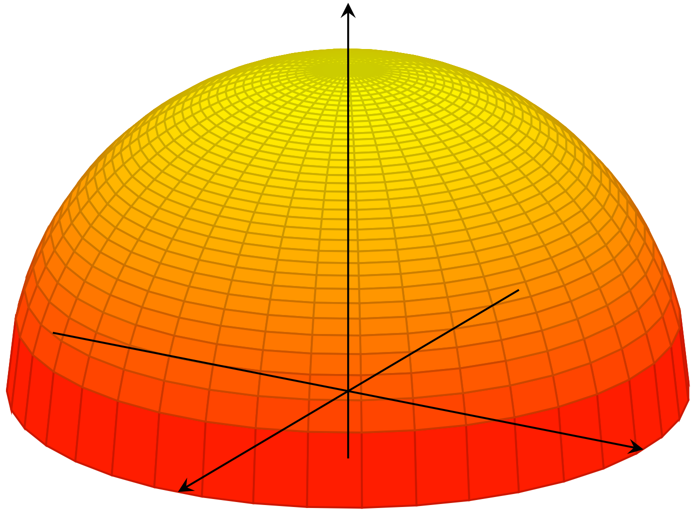
$$f:\ \mathbf{R}^2 \longrightarrow \mathbf{R},\ f(x,y) = 3x - y$$



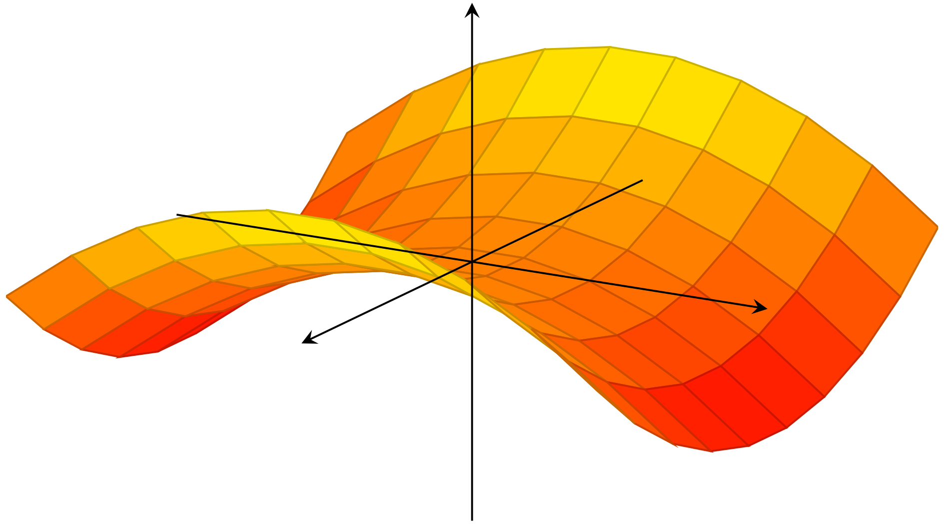
$$f:\ \mathbf{R}^2 \longrightarrow \mathbf{R},\ f(x,y) = 9 - x^2 - y^2$$



$$f:\ \mathbf{B}^1 \longrightarrow \mathbf{R},\ f(x,y) = \sqrt{1 - x^2 - y^2}$$



$$f:\ \mathbf{R}^2 \longrightarrow \mathbf{R},\ f(x,y) = x^2 - y^2$$



6.2 Partial derivatives

Let $f: D \subseteq \mathbf{R}^2 \longrightarrow \mathbf{R}$ and $(x_0, y_0) \in D$.

The partial derivative of f with respect to x at (x_0, y_0) is

$$f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) = \lim_{t \longrightarrow 0} \frac{f(x_0 + t, y_0) - f(x_0, y_0)}{t},$$

if this limit exists.

Example 6.2. Consider $f: \mathbf{R} \times (0, \infty) \longrightarrow \mathbf{R}$ given by $f(x, y) = x \log(y) + xy$.

We can of course (try to) differentiate more than once:

$$
\begin{aligned}
f_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\
f_{xy} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\
f_{yx} &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\
f_{yy} &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)
\end{aligned}
$$

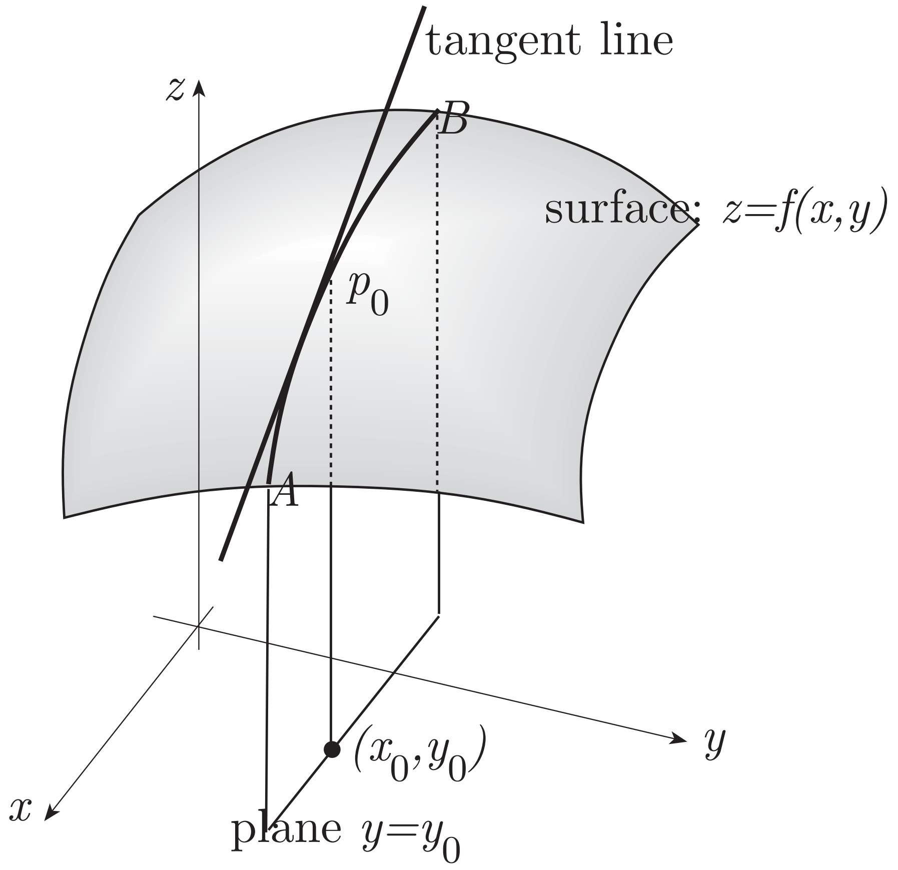
One can show that if the second partial derivatives are continuous, then the mixed partials agree:

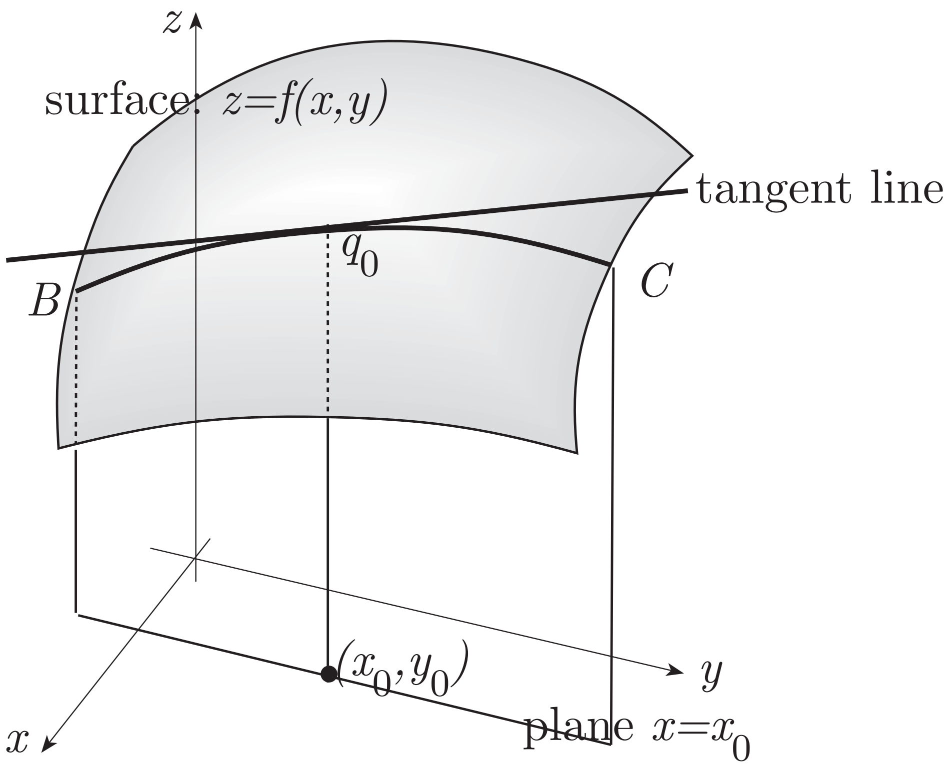
$$f_{xy} = f_{yx}.$$

Example 6.3. Find the second partial derivatives of $f: \mathbf{R} \times (\mathbf{R} \setminus \{0\}) \longrightarrow \mathbf{R}$ given by

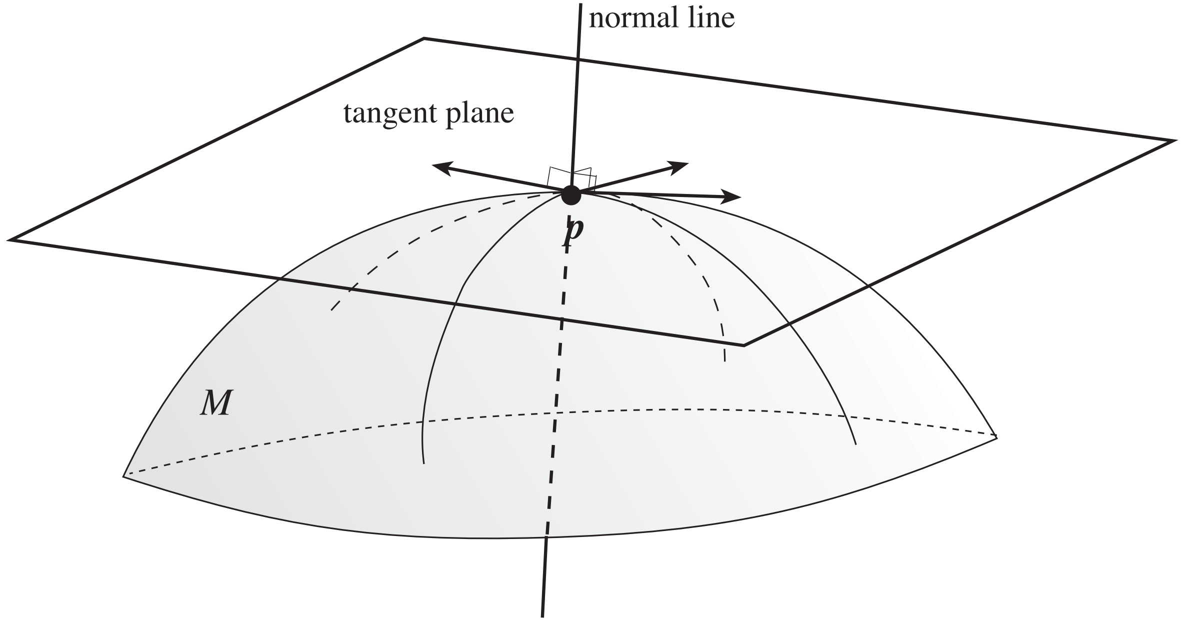
$$f(x, y) = x^3 e^{-2y} + y^{-2} \cos(x).$$

Geometric meaning of partial derivatives





The slope of the tangent line on the left is $f_x(x_0, y_0)$; on the right, $f_y(x_0, y_0)$.



We say that $f : D \subseteq \mathbf{R}^2 \longrightarrow \mathbf{R}$ *is differentiable at a point* (x_0, y_0) *in the interior of* D if the tangent lines to all the curves on the surface $z = f(x, y)$ passing through the point (x_0, y_0) form a plane.

This is called the *tangent plane to the surface at* (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$.

The line orthogonal to the tangent plane and passing through (x_0, y_0, z_0) is called the *normal line to the surface at* (x_0, y_0, z_0).

Theorem 6.4. *If there exists an open ball $B \subseteq D$ containing (x_0, y_0) such that f_x and f_y exist and are continuous at all the points of B, then f is differentiable at (x_0, y_0).*

The equation of the tangent plane is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The plane passes through the point $\big(x_0, y_0, f(x_0, y_0)\big)$ and has normal vector $\begin{bmatrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \\ -1 \end{bmatrix}$.

Example 6.5. Find the Cartesian equation of the tangent plane to the surface

$$z = 1 - x^2 - y^2$$

at the point $(1, 2, -4)$.

At points (x, y) close to (x_0, y_0) we can estimate the value $z = f(x, y)$ using the *linear approximation to f near (x_0, y_0)*:

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example 6.6. Estimate the value $f(0.01, 0.02)$, where

$$f(x, y) = \sqrt{1 - x + 2y}.$$

Chain rule

Suppose $z = f(x, y)$ is a differentiable function of two variables, $x = g(t)$ and $y = h(t)$ are differentiable functions of a single variable t.

Then the function of one variable $z = f(g(t), h(t))$ is differentiable and its derivative is given by the chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Example 6.7. If $z = x^2 - y^2$, $x = \sin(t)$, and $y = \cos(t)$, find $\frac{dz}{dt}$ at $t = \frac{\pi}{3}$.

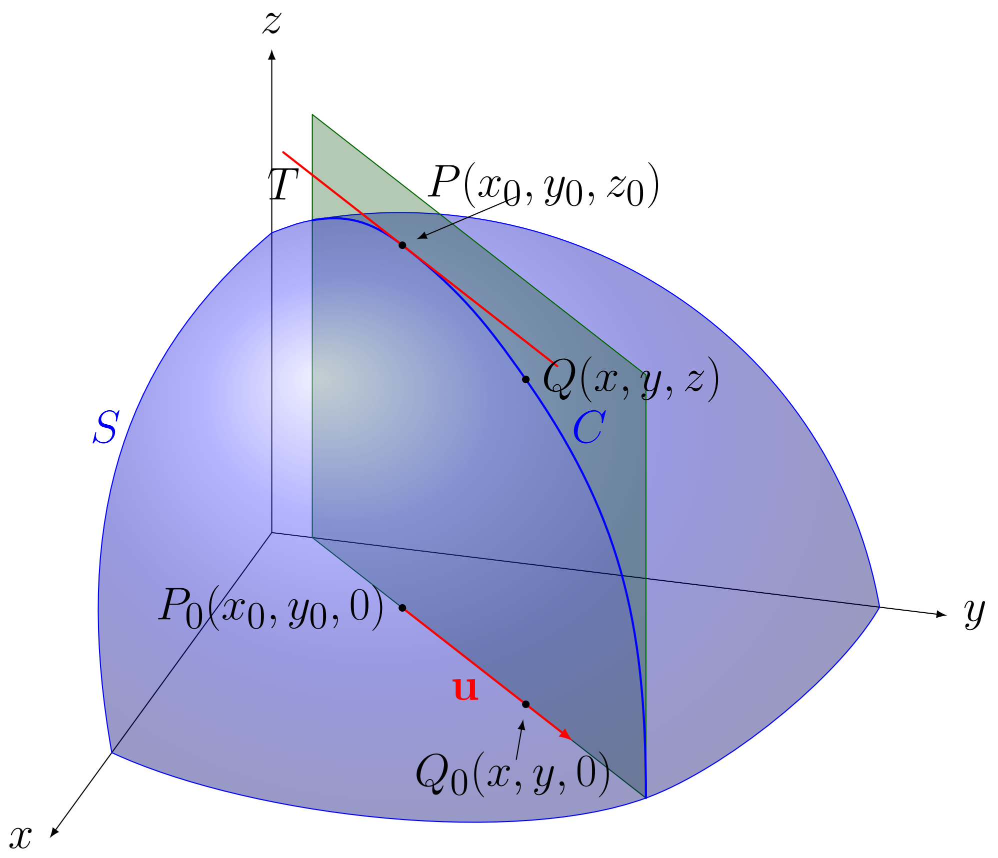
6.3 Directional derivative and gradient

It is useful to know the rate of change of a function in a particular direction of interest.

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ be a **unit vector** in $\mathbf{R}^2$.

The *directional derivative of f in the direction of $\mathbf{u}$ at the point $P_0 = (x_0, y_0)$* is

$$(D_{\mathbf{u}} f)(x_0, y_0) = \frac{d}{dt} f\left(P_0 + t\mathbf{u}\right)\Big|_{t=0}.$$



The *gradient of f* is defined to be

$$\nabla f = \begin{bmatrix} f_x \\ f_y \end{bmatrix}.$$

So the result of the calculation we performed above can be written

$$(D_{\mathbf{u}} f)(P_0) = (\nabla f)(P_0) \cdot \mathbf{u}.$$

Example 6.8. Find the rate of change of the function $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$

$$f(x,y) = 1 - \frac{x^2}{4} - \frac{y^2}{4}$$

at the point $(1, 0)$ in the direction of the vectors

(a) $\mathbf{a} =$ the unit vector making an angle of $\pi/4$ with the x-axis

(b) $\mathbf{b} = \mathbf{e}_2$.

We've seen that if $\mathbf{u}$ is a unit vector, then

$$(D_{\mathbf{u}}f)(P) =$$

What is the direction $\mathbf{u}$ in which the derivative is

- largest

- smallest

Example 6.9. In which directions does $f = xy^2$ increase, resp. decrease, most rapidly at $(1, 2)$?

6.4 Optimisation in two variables

Consider $f:\ D \subseteq \mathbf{R}^2 \longrightarrow \mathbf{R}$ and let (x_0, y_0) be a point in the interior of D.

We say that *f has a local maximum at (x_0, y_0)* if

We say that *f has a local minimum at (x_0, y_0)* if

Theorem 6.10. *If f has a local maximum or local minimum at (x_0, y_0), then either*

- *(x_0, y_0) is a stationary point:* $(\nabla f)(x_0, y_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;
- *or at least one of f_x, f_y does not exist at (x_0, y_0).*

Be careful! Not all stationary points are local maxima or minima.

Example 6.11. Find the stationary points of $f(x, y) = x^2 + y^2$, $g(x, y) = y^2 - x^2$, and $h(x, y) = x^2 + 6xy + 4y^2 + 2x - 4y$.

Suppose f has continuous second order partial derivatives in a neighbourhood of (x_0, y_0).

The symmetric 2×2 matrix

$$\mathbf{H} = \mathbf{H}_f(x_0, y_0) = \begin{bmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}$$

is called *the Hessian of f at (x_0, y_0)*.

Note that we know that $\mathbf{H}$ has real eigenvalues (because it is symmetric).

Theorem 6.12 (Second Derivative Test)**.** *Let* (x_0, y_0) *be a stationary point of* f*. Then* (x_0, y_0) *is a*

(a) *local minimum if* $\mathbf{H}$ *has positive eigenvalues*

(b) *local maximum if* $\mathbf{H}$ *has negative eigenvalues*

(c) *saddle point if* $\mathbf{H}$ *has one positive eigenvalue and one negative eigenvalue.*

The test is inconclusive if $\mathbf{H}$ *is not invertible.*

Example 6.13. Classify the stationary points of $f(x, y) = x^2 + 6xy + 4y^2 + 2x - 4y$.

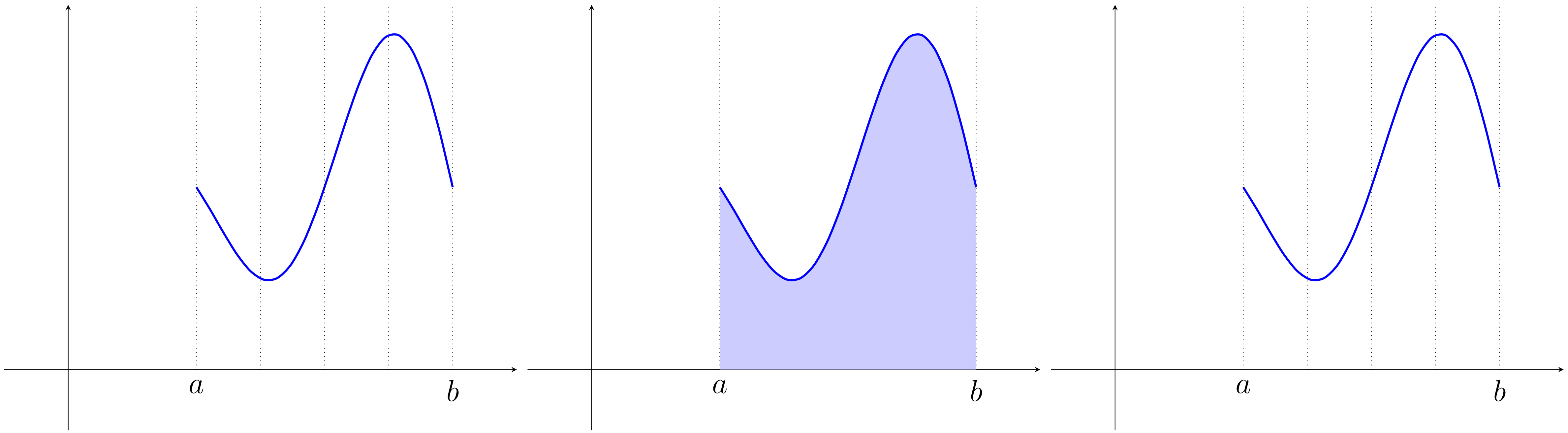
Example 6.14. Classify the stationary points of $f(x, y) = x^2y + x^4 - y^3/3$.

6.5 Double integrals

If $f : (a, b) \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ is a function of a single variable, then the definite integral

$$\int_a^b f(x)\,dx$$

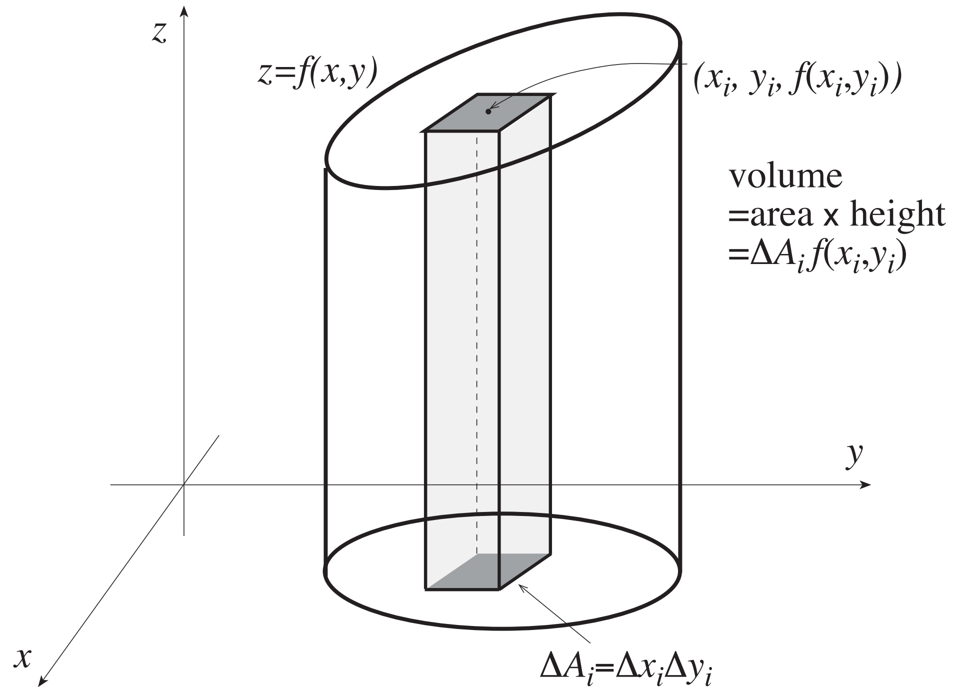
is the (signed) area of the plane region between the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$.



Suppose we are now working with a function $f: \ D \subseteq \mathbf{R}^2 \longrightarrow \mathbf{R}$. We define the *(double) integral of f over D*, denoted

$$\iint_D f(x, y)\, dA,$$

to be the (signed) volume of the solid region between the graph of f, the xy-plane, and the "cylinder over D":



Of course, almost nobody computes actual definite integrals (whether simple or double) via Riemann sums.

In practice, we proceed in a manner similar to partial differentiation: we find the antiderivative with respect to one variable at a time.

Example 6.15.

$$\int (3x^2y + 12xy)\,dx =$$

$$\int (3x^2y + 12xy)\,dy =$$

The simplest type of region for integration is a rectangle:

Theorem 6.16 (Fubini)**.** *If* $D = [a, b] \times [c, d]$ *then*

$$\iint_D f(x, y)\, dA = \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y)\, dx\, dy = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x, y)\, dy\, dx.$$

Example 6.17. Integrate in two ways, changing the order of integration:

$$\iint_{[0,1]\times[0,1]} \left(3x^2y + 12xy\right)\, dA =$$

We've already seen that certain volumes can be expressed as double integrals. There are many other applications:

- If $D \subseteq \mathbf{R}^2$ is a plane region, then its area is given by

- If a flat object occupies a plane region $D \subseteq \mathbf{R}^2$ and $\rho : D \longrightarrow \mathbf{R}_{\geqslant 0}$ is the function describing the density of the object at various points, then the mass of the object is given by

- The coordinates (x_c, y_c) of the centre of mass of a flat object as above are given by

What if D is **not** a rectangle? We consider the cases where $D \subseteq \mathbf{R}^2$ is

(a) bounded by two vertical lines and the graphs of two functions of x:

$$D = \{(x,y) \in \mathbf{R}^2 : \ a \leqslant x \leqslant b, \quad g_1(x) \leqslant y \leqslant g_2(x)\},$$

then

$$\iint_D f(x,y)\,dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx;$$

(b) bounded by two horizontal lines and the graphs of two functions of y:

$$D = \{(x,y) \in \mathbf{R}^2 : \ c \leqslant y \leqslant d, \quad h_1(y) \leqslant x \leqslant h_2(y)\},$$

then

$$\iint_D f(x,y)\,dA = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy.$$

Many bounded domains can be broken up into pieces of the types described above and integrated over each piece.

Example 6.18. Find the centre of mass of a uniformly-dense flat object occupying the plane region D between the parabolas $y^2 = x$ and $x^2 = y$.