4 Inner product spaces

Dot product in $\mathbb{R}^n$

Exercise 4.1. For the given vectors $\mathbf{a}$ and $\mathbf{b}$ find $\mathbf{a}+\mathbf{b}$, $5\mathbf{a}-4\mathbf{b}$, $\mathbf{a}\cdot\mathbf{b}$, $\|\mathbf{a}\|$, and $\|\mathbf{a}-\mathbf{b}\|$.

(a) $\mathbf{a} = \begin{bmatrix} -2 \\ 6 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ -3 \\ -1 \end{bmatrix}$;

(b) $\mathbf{a} = 3\mathbf{e}_1 - 4\mathbf{e}_2 + 21\mathbf{e}_3,\ \mathbf{b} = \mathbf{e}_1 + 2\mathbf{e}_2 - 5\mathbf{e}_3$;

(c) $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_2,\ \mathbf{b} = -\mathbf{e}_2 + \mathbf{e}_3$.

Exercise 4.2. Find the following dot products:

(a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$;

(b) $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$;

(c) $\begin{bmatrix} \sqrt{2} \\ \pi \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{2} \\ -2 \\ 3 \end{bmatrix}$.

Exercise 4.3. Find the angle between the following pairs of vectors:

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$;

(b) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$;

(c) $\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

Exercise 4.4. Let $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

(a) Find the cosine of θ, the angle between $\mathbf{a}$ and $\mathbf{b}$.

(b) Find the distance between the points $(1, 2, 5)$ and $(-3, 1, 0)$.

Exercise 4.5. If $\mathbf{a} = 2\mathbf{e}_1 + x\mathbf{e}_2 + \mathbf{e}_3$ and $\mathbf{b} = 4\mathbf{e}_1 - 2\mathbf{e}_2 - 2\mathbf{e}_3$, find x such that $\mathbf{a}$ is orthogonal to $\mathbf{b}$. Can you find a value of x so that $\mathbf{a}$ and $\mathbf{b}$ are parallel?

Exercise 4.6. If $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{x} + (\mathbf{x} \cdot \mathbf{b})\mathbf{a} = \mathbf{b}$, find the vector $\mathbf{x}$.

Definitions and examples

Exercise 4.7. In $\mathbf{R}^2$, for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, define $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 3x_2 y_2$.

Show that $\langle \mathbf{x}, \mathbf{y} \rangle$ is an inner product on $\mathbf{R}^2$.

Exercise 4.8. In $\mathbf{R}^2$, let $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_2 y_2$. Is this an inner product? If not, why not?

Exercise 4.9. Verify that the operation

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3x_2 y_2$$

gives an inner product in $\mathbf{R}^2$.

Exercise 4.10. Decide which of the suggested operations on $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ in $\mathbf{R}^3$ define an inner product:

(a) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 2x_2 y_2 + x_3 y_3$;

(b) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2$;

(c) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_2 y_2 + x_3 y_3$;

(d) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2$.

Exercise 4.11. Decide which of the operations $\langle p, q \rangle$ on real polynomials $p(x) = a_0 + a_1 x + a_2 x^2$ and $q(x) = b_0 + b_1 x + b_2 x^2$ define an inner product:

(a) $\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$;

(b) $\langle p, q \rangle = a_0 b_0$;

(c) $\langle p, q \rangle = \int_0^1 p(x) q(x)\, dx$.

Exercise 4.12. If $p = a_0 + a_1 x + a_2 x^2$ and $q = b_0 + b_1 x + b_2 x^2$ are any two polynomials in $\mathcal{P}_2$, then

$$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$$

is an inner product on $\mathcal{P}_2$.

(a) Compute $\langle p, q\rangle$ if $p = -2 + x + 3x^2$ and $q = 4 - 7x^2$.

(b) If $p = -2 + 3x + 2x^2$, find $\|p\|$.

(c) If $p = 3 - x + x^2$, $q = 2 + 5x^2$, find $d(p, q)$.

Exercise 4.13. If $U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ and $V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$ are any two 2×2 matrices, then

$$\langle U, V\rangle = u_{11}v_{11} + u_{12}v_{12} + u_{21}v_{21} + u_{22}v_{22}$$

defines an inner product on $M_{2\times 2}$.

(a) Compute $\langle U, V\rangle$ if $U = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$ $V = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$.

(b) If $A = \begin{bmatrix} -2 & 5 \\ 3 & 6 \end{bmatrix}$, find $\|A\|$.

(c) If $A = \begin{bmatrix} 2 & 6 \\ 9 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 7 \\ 1 & 6 \end{bmatrix}$, find $d(A, B)$.

Exercise 4.14. For the vectors $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbf{R}^3$, compute the norms $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ as well as the angle between $\mathbf{x}$ and $\mathbf{y}$ using the following inner products.

(a) $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + x_2y_2 + x_3y_3$;

(b) $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + 3x_2y_2 + x_3y_3$.

Exercise 4.15. Let A be a real invertible $n \times n$ matrix. Show that

$$\langle \mathbf{x}, \mathbf{y}\rangle := \mathbf{y}^T A^T A\mathbf{x} = (A\mathbf{y})^T (A\mathbf{x})$$

defines an inner product in $\mathbf{R}^n$, where $\mathbf{x}$ and $\mathbf{y}$ are column vectors in $\mathbf{R}^n$.

Where do you use the assumption that A is invertible? What happens if it is not invertible?

Exercise 4.16. By choosing appropriate vectors in the Cauchy–Schwarz inequality, prove that

$$(a_1 + \cdots + a_n)^2 \leqslant n(a_1^2 + \cdots + a_n^2)$$

for all real numbers $a_1, \ldots, a_n$. When does equality hold?

Exercise 4.17. Prove that the following holds for all vectors $\mathbf{x}, \mathbf{y}$ in any inner product space V:

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2.$$

Give a geometric interpretation of this result.

[**Hint: consider the parallelogram whose sides are given by the vectors $\mathbf{x}$ and $\mathbf{y}$.**]

Orthogonality and orthonormal bases

Exercise 4.18. In each part determine whether the given vectors are orthogonal with respect to the Euclidean inner product.

(a) $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$;

(b) $\mathbf{u} = \begin{bmatrix} 0 \\ 3 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 5 \\ 2 \\ -1 \\ 0 \end{bmatrix}$.

Exercise 4.19. Let $\mathbf{R}^4$ have the Euclidean inner product, and let $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$. Determine whether the vector $\mathbf{u}$ is orthogonal to the set of vectors

$$W = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} \qquad \text{where} \qquad \mathbf{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 4 \\ 0 \\ 9 \\ 2 \end{bmatrix}.$$

Exercise 4.20. Consider $\mathbf{R}^2$ and $\mathbf{R}^3$ each with the Euclidean inner product. In each part find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

(a) $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$;

(b) $\mathbf{u} = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ -9 \end{bmatrix}$.

Exercise 4.21. Show that $p = 1 - x + 2x^2$ and $q = 2x + x^2$ are orthogonal with respect to the inner product in question 4.**??**.

Exercise 4.22. Let $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. Which of the following matrices are orthogonal to A with respect to the inner product in question 4.**??**?

(a) $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$;

(b) $\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$.

Exercise 4.23. Show that in every inner product space: $\mathbf{v} + \mathbf{w}$ is orthogonal to $\mathbf{v} - \mathbf{w}$ if and only if $\|\mathbf{v}\| = \|\mathbf{w}\|$. Give a geometric interpretation of this result.

Exercise 4.24. Let $\langle , \rangle$ be an inner product on a vector space V, and let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be an orthonormal basis for V.

We proved in the lectures that for each $\mathbf{x} \in V$:

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{x}, \mathbf{u}_2 \rangle \mathbf{u}_2 + \cdots + \langle \mathbf{x}, \mathbf{u}_n \rangle \mathbf{u}_n.$$

Show that

(a) $\langle \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n, \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \cdots + \beta_n \mathbf{u}_n \rangle = \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n$;

(b) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{u}_1 \rangle \langle \mathbf{y}, \mathbf{u}_1 \rangle + \cdots + \langle \mathbf{x}, \mathbf{u}_n \rangle \langle \mathbf{y}, \mathbf{u}_n \rangle$.

Exercise 4.25. For each vector $\mathbf{x}$ and $\mathbf{y}$, give the coordinate vector with respect to the following orthonormal basis (for the dot product):

$$\left\{ \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} \right\}.$$

(a) $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$;

(b) $\mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Exercise 4.26. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be orthogonal vectors in an inner product space V and let $\|\mathbf{x}\|$ be the norm induced by the inner product on V. Show that

$$\|\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_n\|^2 = \|\mathbf{u}_1\|^2 + \|\mathbf{u}_2\|^2 + \cdots + \|\mathbf{u}_n\|^2.$$

Exercise 4.27. Use the Gram–Schmidt procedure to construct orthonormal bases for the subspaces of $\mathbf{R}^n$ spanned by the following sets of vectors (using the dot product):

(a) $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$;

(b) $\begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ -2 \end{bmatrix};$

(c) $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ -2 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 2 \\ 6 \\ -4 \end{bmatrix}.$

Exercise 4.28. Find the orthogonal projection of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ onto the subspace of $\mathbf{R}^3$ spanned by the vectors

(a) $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix};$

(b) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$

Exercise 4.29. Let V be a finite dimensional real vector space with inner product $\langle\ ,\ \rangle$, and let W be a subspace of V. Then the **orthogonal complement** of W is defined as

$$W^\perp = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{w} \in W\}.$$

Prove the following:

(a) $W^\perp$ is a subspace of V.

(b) $W \cap W^\perp = \{\mathbf{0}\}$.

(c) $\dim W + \dim W^\perp = \dim V$.

Exercise 4.30. Let $\mathcal{P}_2$ be the vector space of polynomials of degree at most two with the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

Obtain an orthonormal basis for $\mathcal{P}_2$ from the basis $\{1, x, x^2\}$ using the Gram–Schmidt process.

Exercise 4.31. Show that the infinite set

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin nx, \frac{1}{\sqrt{\pi}} \cos nx : \ n = 1, 2, \ldots \right\}$$

is an orthonormal set in the vector space $C[0, 2\pi]$ of real continuous functions on the interval $[0, 2\pi]$ equipped with the inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(x) g(x) \, dx.$$

Orthogonal matrices

Exercise 4.32. Determine whether or not the given matrix A is orthogonal.

(a) $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$;

(b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Exercise 4.33. Show that the rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Exercise 4.34. For each symmetric matrix A below find a decomposition $A = PDP^T$, where P is orthogonal and D diagonal.

(a) $\begin{bmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}$;

(b) $\begin{bmatrix} -2 & 0 & -36 \\ 0 & -3 & 0 \\ -36 & 0 & -23 \end{bmatrix}$;

(c) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

(d) $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$.

Exercise 4.35. Let A be an orthogonal matrix. Show that $\det A = \pm 1$.

Exercise 4.36. Prove that if A, B are orthogonal $n \times n$ matrices, then so are A^{-1} and AB.

Answers

Solution 4.1. (a) $\begin{bmatrix}1\\3\\0\end{bmatrix}$; $\begin{bmatrix}-22\\42\\9\end{bmatrix}$; -25; $\sqrt{41}$; $\sqrt{110}$.

(b) $\begin{bmatrix}4\\-2\\16\end{bmatrix}$; $\begin{bmatrix}11\\-28\\125\end{bmatrix}$; -110; $\sqrt{466}$; $2\sqrt{179}$.

(c) $\begin{bmatrix}1\\0\\1\end{bmatrix}$; $\begin{bmatrix}5\\9\\-4\end{bmatrix}$; -1; $\sqrt{2}$; $\sqrt{6}$.

Solution 4.2. 0, 6, $5-2\pi$.

Solution 4.3. $\frac{\pi}{2}$, $\frac{2\pi}{3}$, $\arccos\left(\frac{1}{\sqrt{15}}\right)$.

Solution 4.4. (a) $\cos\theta = \frac{-1}{10\sqrt{3}}$;

(b) $d\big((1,2,5),(-3,1,0)\big) = \left\| \begin{bmatrix}-3\\1\\0\end{bmatrix} - \begin{bmatrix}1\\2\\5\end{bmatrix} \right\| = \sqrt{42}$.

Solution 4.5. To be orthogonal: $x = 3$; to be parallel: no solution.

Solution 4.6. $\mathbf{x} = \mathbf{b} - \|\mathbf{b}\|^2\mathbf{a}$.

Solution 4.7. Just like the first example of inner product in the lecture notes.

Solution 4.8. No. Note that $\langle \mathbf{e}_2, \mathbf{e}_2\rangle = -1 < 0$. Alternatively, note that $\langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\rangle = 0$.

Solution 4.9. Everything is routine until you get to axioms 4(a) and 4(b). We have

$$\langle \mathbf{x}, \mathbf{x}\rangle = x_1^2 - 2x_1x_2 + 3x_2^2 = (x_1 - x_2)^2 + 2x_2^2.$$

Now this is clearly non-negative (4(a)), and it is zero if and only if both squares are zero, which imply that $x_2 = 0$ and $x_1 = 0$, whence 4(b).

Solution 4.10. Yes, no, no, no.

Solution 4.11. Yes, no, yes.

Solution 4.12. (a) -29.

(b) $\sqrt{17}$.

(c) $3\sqrt{2}$.

Solution 4.13. (a) 3.

(b) $\sqrt{74}$.

(c) $\sqrt{105}$.

Solution 4.14. (a) $\|\mathbf{x}\| = \sqrt{2}, \quad \|\mathbf{y}\| = 1, \quad \theta = \frac{\pi}{4}$;

(b) $\|\mathbf{x}\| = 2, \quad \|\mathbf{y}\| = \sqrt{3}, \quad \theta = \frac{1}{6}\pi$.

Solution 4.15. Most of this is a straightforward verification of the inner product axioms.

The interesting part is axiom 4(b), where the invertibility of A is crucial, as you want, from $A\mathbf{x} = \mathbf{0}$, to deduce that $\mathbf{x} = \mathbf{0}$.

Solution 4.16. Pattern-match the inequality that you want to prove to the statement of Cauchy–Schwarz. One good possibility is $\mathbf{u} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, which indeed gives us what we want.

Equality holds iff $\mathbf{v} = \lambda\mathbf{u}$.

Solution 4.17. Simply apply $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$.

Geometric interpretation: the sum of the squares of the lengths of the two diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

Solution 4.18. Yes, no.

Solution 4.19. No.

Solution 4.20. $-\dfrac{1}{\sqrt{2}}$; 0.

Solution 4.21. Show $\langle p, q \rangle = 0$.

Solution 4.22. Yes, no.

Solution 4.23. We have

$$0 = \langle \mathbf{v} + \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle - \langle \mathbf{w}, \mathbf{w} \rangle = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2.$$

For a geometric interpretation, draw the parallelogram formed by two vectors $\mathbf{v}$ and $\mathbf{w}$ and consider the diagonals.

Solution 4.24. Use the properties of the inner product and the orthonormality of the basis.

Solution 4.25. (a) $[\mathbf{x}] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$;

(b) $[\mathbf{y}] = \begin{bmatrix} 1/3 \\ 4/3 \\ -1/3 \end{bmatrix}$.

Solution 4.26. The case $n = 1$ is trivial. The case $n = 2$ is the usual Pythagoras theorem.

Use induction on n to do the general case.

Solution 4.27. (a) $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \quad \frac{1}{\sqrt{10}} \begin{bmatrix} 2 \\ -1 \\ -2 \\ -1 \end{bmatrix}$;

(b) $\frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad \frac{1}{3} \begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \quad \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$;

(c) $\frac{1}{4} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ – the space has dimension 2.

Solution 4.28. (a) $\dfrac{1}{9} \begin{bmatrix} 5x - 2y + 4z \\ -2x + 8y + 2z \\ 4x + 2y + 5z \end{bmatrix}$;

(b) $\dfrac{1}{14} \begin{bmatrix} 5x + 6y + 3z \\ 6x + 10y - 2z \\ 3x + -2y + 13z \end{bmatrix}$.

Solution 4.29. (a) We apply the Subspace Theorem:

- For any $\mathbf{w} \in W$ we have $\langle \mathbf{0}, \mathbf{w} \rangle = 0$, therefore $\mathbf{0} \in W^{\perp}$.
- If $\mathbf{v}_1, \mathbf{v}_2 \in W^{\perp}$ then $\langle \mathbf{v}_1, \mathbf{w} \rangle = \langle \mathbf{v}_2, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$. Therefore $\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$, hence $\mathbf{v}_1 + \mathbf{v}_2 \in W^{\perp}$.
- If $\lambda \in \mathbf{R}$ and $\mathbf{v} \in W^{\perp}$, then $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$, so $\langle \lambda\mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$, so $\lambda\mathbf{v} \in W^{\perp}$.

(b) Let $\mathbf{v} \in W \cap W^\perp$, then $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$. But $\mathbf{v} \in W$, so $\langle \mathbf{v}, \mathbf{v} \rangle = 0$, hence $\mathbf{v} = \mathbf{0}$.

(c) Let $\mathcal{B}_1 = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ be an orthonormal basis of W. Extend it to an orthonormal basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ of V, with $\mathcal{B}_2 = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$. Every vector in $\mathcal{B}_2$ is orthogonal to each vector in $\mathcal{B}_1$, hence to each vector in $W = \text{Span}(\mathcal{B}_1)$, so $\mathcal{B}_2 \subseteq W^\perp$.

Let $\mathbf{v} \in W^\perp$. Since $\mathbf{v} \in V$, we have

$$\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n + b_1\mathbf{w}_1 + \cdots + b_m\mathbf{w}_m,$$

where $a_i = \langle \mathbf{v}, \mathbf{u}_i \rangle$ and $b_i = \langle \mathbf{v}, \mathbf{w}_i \rangle = 0$. Therefore $\mathbf{v} \in \text{Span}(\mathcal{B}_2)$.

We conclude that $\mathcal{B}_2$ is a basis for $W^\perp$, and we are done.

Solution 4.30. $\dfrac{1}{\sqrt{2}}, \ \dfrac{\sqrt{3}}{\sqrt{2}}x, \ \dfrac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \dfrac{1}{3}\right).$

Solution 4.31. It comes down to some integrals:

$$\int_0^{2\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_0^{2\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_0^{2\pi} \cos(mx)\sin(nx)\,dx = 0$$

Solution 4.32. Orthogonal, not orthogonal.

Solution 4.33. Multiply A^TA and use the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$.

Solution 4.34. (a) $P = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$

(b) $P = \frac{1}{5}\begin{bmatrix} 0 & 4 & 3 \\ 5 & 0 & 0 \\ 0 & -3 & 4 \end{bmatrix}, \qquad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & -50 \end{bmatrix}.$

(c) $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

(d) $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$

Solution 4.35. We have

$$\det(A^T A) = \det(I) = 1,$$

but $\det(A^T) = \det(A)$ so we conclude that $\det(A)^2 = 1$, so $\det(A) = \pm 1$.

Solution 4.36. We have $A^T A = I$, so A and A^T are inverses of each other.

Therefore

$$(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = AA^{-1} = I,$$

hence A^{-1} is orthogonal.

For AB, we have

$$(AB)^T (AB) = B^T A^T AB = B^T IB = B^T B = I.$$