2 Matrices and linear equations

Arithmetic of matrices

Exercise 2.1. Evaluate the following matrix products:

(a) $\begin{bmatrix} 3 & 4 & 2 \\ 1 & 3 & 6 \\ 7 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}$;

(b) $\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 \end{bmatrix}$;

(c) $\begin{bmatrix} 7+i & 6 & -4+3i \end{bmatrix} \begin{bmatrix} 0 \\ 1-i \\ 1 \end{bmatrix}$;

(d) $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 7 & 6 & -4 \end{bmatrix}$;

(e) $\begin{bmatrix} 3 & -6 & 0 \\ 0 & 2 & -2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$;

(f) $\begin{bmatrix} 4 & 3 \\ 6 & 6 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$.

Exercise 2.2. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ -1 & 0 & -2 \end{bmatrix}.$$

Compute the matrix

$$B = -\frac{1}{3}(A^2 - 2A - 4I)$$

and show that it satisfies

$$BA = AB = I.$$

Exercise 2.3. Verify that

$$\begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}.$$

(Does this seem pointless? That's because we haven't discussed the matrix of a rotation yet, at which point this exercise will be more interesting.)

The *transpose* of an $m \times n$ matrix $A = [a_{ij}]$ is the $n \times m$ matrix $A^T = [a_{ji}]$ obtained by interchanging the rows and columns of A. For instance:

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 7 & -2 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 7 \\ 2 & -2 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Exercise 2.4. Prove that for any matrices A and B of the same size, and any scalar λ, the following hold:

(a) $\left(A^T\right)^T = A$;

(b) $\left(\lambda A\right)^T = \lambda\left(A^T\right)$;

(c) $\left(A + B\right)^T = A^T + B^T$;

(d) $\left(AB\right)^T = B^T A^T$.

Exercise 2.5. Write down the 3×2 matrices A and B which have entries given by $A_{ij} = i + j$ and $B_{ij} = (-1)^{i+j}$. Calculate $A^T B$.

Exercise 2.6. Give 3×3 examples of the following:

(a) a diagonal matrix; that is, a matrix A with $A_{ij} = 0$ if $i \neq j$;

(b) a scalar matrix; that is, a diagonal matrix A with $A_{ii} = A_{jj}$;

(c) a symmetric matrix; that is, a matrix A with $A_{ij} = A_{ji}$;

(d) an upper triangular matrix; that is, a matrix A with $A_{ij} = 0$ if $i > j$.

Exercise 2.7. Let A and B be $n \times n$ matrices and suppose that $A^2 = A$. Show that $(AB - ABA)^2 = 0$.

Exercise 2.8. Suppose that a 2×2 matrix A satisfies $AB = BA$ for every 2×2 matrix B. Show that A must be a scalar matrix (that is, a scalar multiple of the identity matrix).

Hint: Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and find the conditions imposed on a, b, c, d by the equality $AB = BA$ when $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Gaussian elimination and echelon forms

Exercise 2.9. Following the algorithm, reduce the following matrices to row echelon form (which is not unique), and then to reduced row echelon form (which is unique). Keep a precise record of the elementary row operations you use. As usual, $i \in \mathbf{C}$ denotes a square root of -1.

(a) $\begin{bmatrix} 4 & -8 & 16 \\ 1 & -3 & 6 \\ 2 & 1 & 1 \end{bmatrix}$;

(b) $\begin{bmatrix} 2+i & 2+i & 5 & 6+i \\ 1-2i & 1-2i & -2+i & 2-i \end{bmatrix}$;

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 2 \\ 0 & 2 \end{bmatrix}$;

(d) $\begin{bmatrix} 0 & 2 & 1 & 4 \\ 0 & 0 & 2 & 6 \\ 1 & 0 & -3 & 2 \end{bmatrix}$;

(e) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 1 & 1 \end{bmatrix}$;

(f) $\begin{bmatrix} 0 & 0 & 2 & 7 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & -4 & 5 \\ -2 & 2 & -5 & 4 \end{bmatrix}$.

Exercise 2.10. What is the rank of the $m \times n$ matrix with 1 in every position? What is the rank of the $m \times n$ *chessboard matrix* with (i, j) entry

$$a_{ij} = \begin{cases} 0 & \text{if } i+j \text{ is even} \\ 1 & \text{if } i+j \text{ is odd?} \end{cases}$$

Linear equations

Exercise 2.11. Which of the following systems of equations are linear?

(a) $$\begin{aligned} x_1 - 3x_2 \quad\quad &= x_3 - 4 \\ x_4 \quad\quad &= 1 - x_1 \\ x_1 + \; x_4 + x_3 - 2 &= 0 \end{aligned}$$

(b) $$\begin{aligned} 3x - xy \quad &= 1 \\ x + 2xy - y &= 0 \end{aligned}$$

(c) $$\begin{aligned} y &= x - 1 \\ x &= 1 - y \end{aligned}$$

(d) $$\begin{aligned} x^2 &= y \\ x + y &= 1 \end{aligned}$$

Exercise 2.12. For each of the matrices from Question 2.8 write down the corresponding homogeneous system and the rank of the matrix, and solve the system of equations from the reduced row echelon form. **Note**: Elementary row operations leave columns with all zeros unchanged.

Exercise 2.13. Using Gaussian elimination, solve the following homogeneous systems:

(a) $$\begin{array}{rcl} (2+i)x - iy + (3-i)z + \qquad u &=& 0 \\ x + y \qquad\qquad - (1-i)u &=& 0 \end{array}$$

(b) $$\begin{array}{rcl} 7x + 7y - 16z + 6w &=& 0 \\ 4x - y + 3z + 7w &=& 0 \\ 3x + y - 2z + 4w &=& 0 \end{array}$$

Exercise 2.14. Use row reduction to find the ranks of the coefficient and augmented matrices of the following systems of equations. Decide whether the system has

i. no solution vector,

ii. a unique solution vector,

iii. more than one solution vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Solve the system where possible and describe the **system** geometrically.

(a) $$\begin{array}{rcr} 3x - 2y + 4z &=& 3 \\ x - y + z &=& 7 \\ 4x - 3y + 5z &=& 1 \end{array}$$

(b) $$\begin{array}{rcr} x + 2y - z &=& -1 \\ 2x + 7y - z &=& 3 \\ -3x - 12y + z &=& 0 \end{array}$$

(c) $$\begin{array}{rcr} 3x - 4y + z &=& 2 \\ -5x + 6y + 10z &=& 7 \\ 8x - 10y - 9z &=& -5 \end{array}$$

(d)
$$\begin{aligned} 2x - 3y + 5z &= 10 \\ 4x + 7y - 2z &= -5 \\ 2x - 4y + 25z &= 31 \end{aligned}$$

Exercise 2.15. Using Gaussian elimination, solve the following system of equations:

$$\begin{aligned} 2x_1 + x_2 + 3x_3 + x_4 &= 3 \\ x_1 + x_2 + x_3 - x_4 &= 6 \\ x_1 - x_2 + 3x_3 + 5x_4 &= -12 \\ 4x_1 + x_2 + 7x_3 + 5x_4 &= -3 \end{aligned}$$

Exercise 2.16. Determine the values of k for which the system of linear equations has

i. no solution vector,

ii. a unique solution vector,

iii. more than one solution vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(a)
$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= 1 \\ x + y + kz &= 1 \end{aligned}$$

(b)
$$\begin{aligned} 2x + (k-4)y + (3-k)z &= 1 \\ 2y + (k-3)z &= 2 \\ x - 2y + z &= 1 \end{aligned}$$

(c)
$$\begin{aligned} x + 2y + kz &= 1 \\ 2x + ky + 8z &= 3 \end{aligned}$$

(d)
$$\begin{aligned} x - 3z &= -3 \\ 2x + ky - z &= -2 \\ x + 2y + kz &= 1 \end{aligned}$$

For the cases ii. and iii. find the solutions.

Exercise 2.17. Determine the conditions on a, b, c so that the system has a solution:

(a)
$$\begin{aligned} x + 2y - 3z &= a \\ 3x - y + 2z &= b \\ x - 5y + 8z &= c \end{aligned}$$

(b)
$$\begin{aligned} x - 2y + 4z &= a \\ 2x + 3y - z &= b \\ 3x + y + 2z &= c \end{aligned}$$

Find the solutions when they exist.

Inverses

Exercise 2.18. Use elementary row operations to find, where possible, the inverses of the following matrices:

(a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 9 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 7 & 2 \\ -3 & 1 & -7 \\ 2 & 4 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 7 & -4 \\ 0 & 9 & -1 \end{bmatrix}$

Exercise 2.19. Use elementary row operations to find, where possible, the inverses of the following matrices:

(a) $\begin{bmatrix} 1+i & i \\ 1 & 1-i \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -i \\ 3+i & 1 \end{bmatrix}$

Exercise 2.20. Find the inverse of the matrix

$$A = \begin{bmatrix} 4 & 3 & -2 & 0 \\ 1 & 2 & -3 & 1 \\ 2 & -1 & 1 & -3 \\ 1 & -3 & 1 & -2 \end{bmatrix}.$$

Hence solve the system

$$\begin{aligned} 4x_1 + 3x_2 - 2x_3 \phantom{{}+ x_4} &= 1 \\ x_1 + 2x_2 - 3x_3 + x_4 &= 0 \\ 2x_1 - x_2 + x_3 - 3x_4 &= -1 \\ x_1 - 3x_2 + x_3 - 2x_4 &= 3 \end{aligned}$$

Check your answer by matrix multiplication.

Determinants

Exercise 2.21. Evaluate the following determinants by Gaussian elimination to triangular form:

(a) $\begin{vmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 3 \end{vmatrix}$

(b) $\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{vmatrix}$

(c) $\begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & 1 \\ 3 & 0 & -3 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 2 \end{vmatrix}$

(e) $\begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & 1 & 2 & 6 \\ 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & -5 \end{vmatrix}$

(f) $\begin{vmatrix} 3 & 9 & 27 & 81 \\ 1 & 1 & 1 & 1 \\ -2 & 4 & -8 & 16 \\ 2 & 4 & 8 & 16 \end{vmatrix}$

Exercise 2.22. Suppose

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 1.$$

Find the following determinants:

(a) $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$

(b) $\begin{vmatrix} a & -b & c \\ d & -e & f \\ g & -h & i \end{vmatrix}$

(c) $\begin{vmatrix} d & e & f \\ 3g & 3h & 3i \\ a & b & c \end{vmatrix}$

(d) $\begin{vmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{vmatrix}$

(e) $\begin{vmatrix} a & b & c \\ d+a & e+b & f+c \\ g-2a & h-2b & i-2c \end{vmatrix}$

Exercise 2.23. Using a suitable determinant find

(a) the area of the parallelogram spanned by the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$,

(b) the volume of the parallelotope spanned by the vectors $\begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(c) the volume of the parallelotope spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

(d) What does your answer to (c) tell you about the three vectors?

Exercise 2.24. Establish the following factorisations:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)$$

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (y-z)(z-x)(x-y)(yz+zx+xy)$$

Exercise 2.25. Evaluate the following determinants:

(a) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

(b) $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$

Exercise 2.26. The famous Vandermonde determinant is defined by

$$D_n = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{vmatrix}$$

Use row operations to evaluate D_2, D_3, and D_4, expressing your answer as a fully factorised expression to exhibit the fact that $D_n = 0$ if any two of the parameters λ_1, λ_2, …, λ_n are equal.

Answers

Solution 2.1. (a) $\begin{bmatrix} 1 & 2 \\ -5 & -1 \\ 7 & 15 \end{bmatrix}$;

(b) matrices incompatible;

(c) $\begin{bmatrix} 2-3i \end{bmatrix}$;

(d) $\begin{bmatrix} 0 & 0 & 0 \\ 7 & 6 & -4 \\ 7 & 6 & -4 \end{bmatrix}$;

(e) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$;

(f) $\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$.

Solution 2.2.

$$B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} \\ -1 & 0 & -1 \end{bmatrix}.$$

Solution 2.3. Straightforward but tedious calculation: compute the matrix product and notice that the entries of the resulting matrix can be rewritten as desired by appealing to the laws for sine and cosine of a sum of angles.

Solution 2.4. (a) Clear from the definition: if we swap rows and columns twice, we end up with the original matrix.

(b) Let $C = \lambda A$, then $c_{ij} = \lambda a_{ij}$. We have

$$(\lambda A)^T = C^T = [c_{ji}] = [\lambda a_{ji}] = \lambda [a_{ji}] = \lambda A.$$

(c) We have

$$(A+B)^T = [a_{ij} + b_{ij}]^T = [a_{ji} + b_{ji}] = [a_{ji}] + [b_{ji}] = A^T + B^T.$$

(d) Let $C = AB$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Therefore

$$(AB)^T = C^T = [c_{ji}] = \Big[\sum_{k=1}^{n} a_{jk}b_{ki}\Big].$$

On the other hand we have

$$B^T A^T = [b_{ji}][a_{ji}] = \Big[\sum_{k=1}^{n} b_{ki}a_{jk}\Big].$$

Solution 2.5.

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad A^T B = \begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix}.$$

Solution 2.6. (a) $\begin{bmatrix} 28 & 0 & 0 \\ 0 & 29 & 0 \\ 0 & 0 & 30 \end{bmatrix}$;

(b) $\begin{bmatrix} 29 & 0 & 0 \\ 0 & 29 & 0 \\ 0 & 0 & 29 \end{bmatrix}$;

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 30 \end{bmatrix}$;

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 30 \end{bmatrix}$.

Solution 2.7. Compute

$$\begin{aligned} (AB - ABA)^2 &= (AB - ABA)(AB - ABA) \\ &= ABAB - ABABA - ABAAB + ABAABA \\ &= ABAB - ABABA - ABAB + ABABA \\ &= 0. \end{aligned}$$

Solution 2.8. Compute the two sides of $AB = BA$ for both matrices B given in the hint. We get the conditions

$$b = 0, \quad c = 0, \quad a = d,$$

which imply that $A = aI$.

Note: The same statement holds for $n \times n$ matrices for any $n \geqslant 2$.

Solution 2.9. A row echelon form and the reduced row echelon form are given. Note that row echelon forms are not unique.

(a) REF: $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, RREF: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

(b) REF: $\begin{bmatrix} 1 & 1 & 2-i & 2.6-0.8i \\ 0 & 0 & 1 & 0.7-0.4i \end{bmatrix}$, RREF: $\begin{bmatrix} 1 & 1 & 0 & 1.6+0.7i \\ 0 & 0 & 1 & 0.7-0.4i \end{bmatrix}$;

(c) REF: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, RREF: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$;

(d) REF: $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$, RREF: $\begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$;

(e) REF: $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, RREF: $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$;

(f) REF: $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, RREF: $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Solution 2.10. For the matrix of ones, the rank is always one regardless of size. For the chessboard matrix, the rank depends on the size:

- if $m = n = 1$, the rank is zero;
- if $m = 1$ and $n \geqslant 2$, the rank is one;
- if $m \geqslant 2$ and $n = 1$, the rank is one;
- in all remaining cases, the rank is two.

Solution 2.11. (a) linear;

(b) nonlinear;

(c) linear;

(d) nonlinear.

Solution 2.12. (a) Rank 3; $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

(b) Rank 2; $\left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1.6 - 0.7i \\ 0 \\ -0.7 + 0.4i \\ 1 \end{bmatrix}, a, b \in \mathbf{R} \right\}$.

(c) Rank 2; $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

(d) Rank 3; $\left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t \begin{bmatrix} -11 \\ -\frac{1}{2} \\ -3 \\ 1 \end{bmatrix}, t \in \mathbf{R} \right\}$.

(e) Rank 3; $\left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, t \in \mathbf{R} \right\}$.

(f) Rank 3; $\left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, t \in \mathbf{R} \right\}$.

Solution 2.13. (a) $\left\{ \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = t \begin{bmatrix} -1 + 2i \\ 1 - 2i \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 + i \\ 3 - 5i \\ 0 \\ 4 \end{bmatrix}, \ s, t \in \mathbf{C} \right\}$;

(b) $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} -1 \\ 17 \\ 7 \\ 0 \end{bmatrix} + b \begin{bmatrix} -11 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \ a, b \in \mathbf{R} \right\}$.

Solution 2.14. (a) and (b) $\operatorname{rank}(A|B) > \operatorname{rank} A$: no solution so the system consists of three planes that do not all intersect in any points.

(c) $\operatorname{rank}(A|B) = \operatorname{rank} A = 2$: multiple solutions: $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 23 \\ \frac{35}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -20 \\ -\frac{31}{2} \\ 0 \end{bmatrix}, \alpha \in \mathbf{R} \right\}$; the system consists of three planes intersecting in a line.

(d) $\operatorname{rank}(A|B) = \operatorname{rank} A = 3$: unique solution: $\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$; the system consists of three planes intersecting in a single point.

Solution 2.15.

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \\ 0 \\ 0 \end{bmatrix} \quad : \quad \alpha, \beta \in \mathbf{R} \right\}.$$

Solution 2.16. (a) i.

$k = -2$: no solution;

b) $k \in \mathbf{R} \setminus \{-2, 1\}$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{k+2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$;

c) $k = 1$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(b) i.

$k = 2$: no solution;

b) $k \in \mathbf{R} \setminus \{-1, 2\}$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{k-2} \begin{bmatrix} k-2 \\ 1 \\ 2 \end{bmatrix}$;

c) $k = -1$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

(c) i.

$k = 4$: no solution;

b) no such k;

c) $k \in \mathbf{R} \setminus \{4\}$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} -k-4 \\ 2 \\ 1 \end{bmatrix} + \dfrac{1}{k-4} \begin{bmatrix} k-6 \\ 1 \\ 0 \end{bmatrix}$.

(d) i.

$k = -5$: no solution;

b) $k \in \mathbf{R} \setminus \{-5, 2\}$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{k+5} \begin{bmatrix} -3k-3 \\ 4 \\ 4 \end{bmatrix}$;

c) $k = 2$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ -\frac{5}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}$.

Solution 2.17. (a) If $c = -2a + b$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 11 \\ 7 \end{bmatrix} + \dfrac{1}{7} \begin{bmatrix} a+2b \\ 3a-b \\ 0 \end{bmatrix}$, $\alpha \in \mathbf{R}$

(b) For all a, b, c: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{7} \begin{bmatrix} -7 & -8 & 10 \\ 7 & 10 & -9 \\ 7 & 7 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Solution 2.18. (a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & -3 \\ 0 & -9 & 4 \end{bmatrix}$

(c) $\dfrac{1}{11} \begin{bmatrix} 29 & 1 & -51 \\ -11 & 0 & 22 \\ -14 & -2 & 25 \end{bmatrix}$

(d) no inverse

Solution 2.19. (a) $\dfrac{1}{5} \begin{bmatrix} 3-i & 1-2i \\ -2-i & 1+3i \end{bmatrix}$

(b) $\dfrac{1}{10} \begin{bmatrix} 1-3i & 3+i \\ -6+8i & 2-6i \end{bmatrix}$

Solution 2.20. $\dfrac{1}{45} \begin{bmatrix} 17 & -11 & -13 & 14 \\ -1 & -2 & 14 & -22 \\ 10 & -25 & -5 & -5 \\ 15 & -15 & -30 & 15 \end{bmatrix}$; $\dfrac{1}{5} \begin{bmatrix} 8 \\ -9 \\ 0 \\ 10 \end{bmatrix}$

Solution 2.21. (a) -5

(b) -9

(c) 0

(d) 1

(e) 2

(f) -1440

Solution 2.22. (a) -1

(b) -1

(c) 3

(d) 8

(e) 1

Solution 2.23. (a) 11

(b) 2

(c) 0

(d) The three vectors in (c) must be coplanar.

Solution 2.24. You can compute each of the determinants by expansion along the first row (for example), then expand the product on the right hand side and compare.

Solution 2.25. (a) $(b-a)(c-a)(c-b)$

(b) $(b-a)(c-a)(c-b)(a+b+c)$

Solution 2.26. $D_2 = \lambda_2 - \lambda_1$, $D_3 = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$ and
$D_4 = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)$.