5 Introduction to multivariable calculus

Partial and directional derivatives, tangent plane

Exercise 5.1. Find the first and second partial derivatives of the following functions:

(a) $f(x, y) = 3x^2 + 2xy + y^5$

(b) $f(x, y) = \sin(x)\sin(y)$

(c) $f(x, y) = (x^3 - y^2)^2$

(d) $f(r, s) = \sqrt{r^2 + s^2}$

(e) $f(x, y) = ye^x$

(f) $f(u, w) = \arctan(u/w)$

(g) $f(r, s, t) = r^2 e^{2s}\cos(t)$.

Exercise 5.2. By using the chain rule find $\dfrac{df}{dt}$ for each of the following:

(a) $f(x, y) = x^2 + y^2$ where $x = 2\sin(t)$ and $y = 3\cos(t)$,

(b) $f(x, y) = x\log(y)$ where $x = t^2 + 1$ and $y = e^t$,

(c) $f(x, y) = e^{xy}$ where $x = 2e^{-t}$ and $y = e^t$.

Exercise 5.3. Suppose $f(x, y) = x^3 + 2xy$ and that $x = t, y = 1 + t^2$.

(a) Express f as a function of t only and then differentiate with respect to t to obtain $\dfrac{df}{dt}$.

(b) Calculate the partial derivatives f_x and f_y and hence evaluate $\dfrac{df}{dt}$.

Exercise 5.4. Find equations of the tangent plane and the normal line to the graph of the given function at the indicated point:

(a) $z = 4x^2 + y^2 - 78$ at $(2, 1, -61)$

(b) $z = x\sin(y)$ at $\left(1, \frac{\pi}{2}, 1\right)$

(c) $z = 4xy$ at $(0, 0, 0)$

(d) $z = 4x^2 - y^2$ at $(5, -8, 36)$

(e) $z = 2e^{-x}\cos(y)$ at $\left(0, \frac{\pi}{3}, 1\right)$.

Exercise 5.5. For the following functions f, find the gradient vector ∇f at the indicated point.

(a) $f(x, y) = x^3 + 2xy + xy^2$ at $(0, 1)$

(b) $f(x, y) = \sin(x)\cos(y)$ at $(\pi, \pi/4)$

(c) $f(x, y) = 3x^2 - xy + y^2$ at $(2, 1)$.

Exercise 5.6. Find the directional derivatives of the following functions at the indicated point in the direction specified.

(a) $f(x, y) = \arcsin(x/y)$ at $(1, 2)$ in the direction $\pi/4$ anticlockwise from the x-axis

(b) $f(x, y) = \sqrt{x^2 + y^2}$ at $(3, 4)$ in the direction away from the origin towards the point $(3, 4)$

(c) $f(x, y) = x^2 - 3xy + y^2$ at $(1, 1)$ in the direction from $(1, 1)$ towards $(1, 2)$

(d) $f(x, y) = x^3y^2$ at $(-1, 2)$ in the direction $\pi/3$ clockwise from the positive x-axis

(e) $g(x, y) = \sin(xy)$ at $(1/6, \pi)$ in the direction of the unit vector $\begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$

(f) $h(x, y) = x^2e^{2y}$ at $(4, 3)$ towards the point $(6, 0)$.

Exercise 5.7. (a) Find the direction in which the function $f(x, y) = x^3 + y^2 - 6xy$ increases and decreases most rapidly at the point $(3, 3)$.

(b) Find the direction in the xy-plane one should travel, starting from the point $(1, 1)$, to obtain the most rapid rate of decrease of $f(x, y) = (x + y - 2)^2 + (3x - y - 6)^2$.

(c) In which direction in the xy-plane is the directional derivative of the function $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ at the point $(1, 1)$ equal to zero?

(d) The temperature at any point (x, y) of a heated plate is $100(x^2 + 2y^2 + 1)^{-1}$. At the point $(1, 2)$, in what direction(s) is the rate of change of temperature (i) zero, and (ii) greatest? How are these directions related?

Exercise 5.8. Check that the level curve $x^3 + 2xy + x^2y^2 + y^3 = 17$ passes through the point $(1, 2)$.

(a) If $f(x, y) = x^3 + 2xy + x^2y^2 + y^3$, calculate ∇f at the point $(1, 2)$ and hence write down the slope of the normal to the curve at $(1, 2)$.

(b) Using implicit differentiation calculate the slope of the above curve at $(1, 2)$.

(c) Check that the product of your answers to parts (a) and (b) is equal to -1.

Exercise 5.9. Find the points on the hyperboloid $x^2 - 2y^2 - 4z^2 = 16$ at which the tangent plane is parallel to the plane $4x - 2y + 4z = 5$.

Stationary points

Exercise 5.10. Find the Hessian matrices for the functions:

(a) $f(x, y) = x^3 + 3xy - y^3$

(b) $f(x, y) = x^4 + y^3 + 32x - 9y$

(c) $f(x, y) = \cos(x) + \cos(y)$

(d) $f(x, y, z) = x^2 + y^2 - 3xy - z$.

Exercise 5.11. (a) Show that $f(x, y) = -4x^2 + xy - y^2$ has a maximum at $(0, 0)$.

(b) Show that $g(x, y) = \dfrac{e^{x+y}}{x^3y^2}$ has a minimum at the point $(3, 2)$.

Exercise 5.12. Find the maxima, minima and saddle points of the following functions:

(a) $x^3 + 3xy - y^3$

(b) $\cos(x) + \cos(y)$.

Exercise 5.13. A company produces two types of surfboard, x thousand of type A and y thousand of type B per year. If the revenue R and cost C equations for the year are (in millions of dollars),

$$R(x, y) = 2x + 3y, \qquad C(x, y) = x^2 - 2xy + 2y^2 + 6x - 9y + 5,$$

determine how many of each type of surfboard should be made per year to maximise the profit. What is the maximum profit?

Double integrals

Exercise 5.14. Find the partial antiderivatives of the function

$$f(x, y) = 3x^2 + 2xy + y^5$$

(a) with respect to x;

(b) with respect to y.

Exercise 5.15. Evaluate the following double integrals where $R = [0, 1] \times [0, 1]$.

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^1 (y \cos x + 2)\, dy dx$

(b) $\displaystyle\int_{-1}^0 \int_1^2 (-xe^y)\, dy dx$

(c) $\displaystyle\iint_R ye^{xy}\,dA$

(d) $\displaystyle\iint_R x^2y^2\cos(x^3)\,dA.$

Exercise 5.16. Using a double integral, find the volume of the solid bounded by the graph of $f(x,y) = 1+2x+3y$, the rectangle $R = [1,2]\times[0,1]$ and the vertical sides of R.

Exercise 5.17. The average value of a continuous function $f(x,y)$ over a rectangle R is defined as

$$f_{\text{average}} = \frac{1}{\text{Area of } R}\iint_R f(x,y)\,dA$$

Find the average value of $f(x,y) = y\sin(xy)$ over the rectangle $[0,1]\times[0,\frac{\pi}{2}]$.

Exercise 5.18. The coordinates $(\bar{x},\bar{y})$ of the centre of mass of a lamina occupying the region D and having a continuous density function $\rho(x,y)$ are

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA \qquad \bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA$$

where the mass is given by

$$m = \iint_D \rho(x,y)\,dA.$$

Find the centre of mass of a rectangular lamina with vertices at $(0,0)$, $(1,0)$, $(1,3)$ and $(0,3)$ if the density function is $\rho = x^2y$.

Answers

Solution 5.1. (a) $f_x = 6x + 2y,\ f_y = 2x + 5y^4,\ f_{xx} = 6,\ f_{yy} = 20y^3,\ f_{xy} = 2,\ f_{yx} = 2.$

(b) $f_x = \cos(x)\sin(y),\ f_y = \sin(x)\cos(y),\ f_{xx} = -\sin(x)\sin(y),$
$f_{yy} = -\sin(x)\sin(y),\ f_{xy} = \cos(x)\cos(y),\ f_{yx} = \cos(x)\cos(y).$

(c) $f_x = 6x^2(x^3 - y^2),\ f_y = -4y(x^3 - y^2),\ f_{xx} = 30x^4 - 12xy^2,$
$f_{yy} = -4x^3 + 12y^2,\ f_{xy} = -12x^2y,\ f_{yx} = -12x^2y.$

(d) $f_r = \dfrac{r}{\sqrt{r^2 + s^2}},\ f_s = \dfrac{s}{\sqrt{r^2 + s^2}},\ f_{rr} = \dfrac{s^2}{(r^2 + s^2)^{3/2}},$
$f_{ss} = \dfrac{r^2}{(r^2 + s^2)^{3/2}},\ f_{rs} = -\dfrac{rs}{(r^2 + s^2)^{3/2}},\ f_{sr} = -\dfrac{rs}{(r^2 + s^2)^{3/2}}.$

(e) $f_x = ye^x,\ f_y = e^x,\ f_{xx} = ye^x,\ f_{yy} = 0,\ f_{xy} = e^x,\ f_{yx} = e^x.$

(f) $f_u = \dfrac{w}{u^2 + w^2},\ f_w = \dfrac{-u}{u^2 + w^2},\ f_{uu} = \dfrac{-2uw}{(u^2 + w^2)^2},$
$f_{ww} = \dfrac{2uw}{(u^2 + w^2)^2},\ f_{uw} = \dfrac{u^2 - w^2}{(u^2 + w^2)^2},\ f_{wu} = \dfrac{u^2 - w^2}{(u^2 + w^2)^2}.$

(g) $f_r = 2re^{2s}\cos t,\ f_s = 2r^2e^{2s}\cos t,\ f_t = -r^2e^{2s}\sin t,\ f_{rr} = 2e^{2s}\cos t,$
$f_{ss} = 4r^2e^{2s}\cos t,\ f_{tt} = -r^2e^{2s}\cos t,\ f_{rs} = 4re^{2s}\cos t,\ f_{rt} = -2e^{2s}\sin t,$
$f_{sr} = 4re^{2s}\cos t,\ f_{st} = -2r^2e^{2s}\sin t,\ f_{tr} = -2e^{2s}\sin t,\ f_{ts} = -2r^2e^{2s}\sin t.$

Solution 5.2. (a)

$$\begin{aligned}
\frac{df}{dt} &= 2x.2\cos(t) + 2y.3(-\sin(t)) \\
&= 4x\cos(t) - 6y\sin(t) \\
\text{or} &= 8\sin(t)\cos(t) - 18\cos(t)\sin(t) \\
&= -10\sin(t)\cos(t)
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{df}{dt} &= \log(y).2t + \frac{x}{y}.e^t = 2t\log(y) + x \\
\text{or} &= 2t^2 + (t^2 + 1) = 3t^2 + 1.
\end{aligned}$$

(c)

$$\begin{aligned}\frac{df}{dt} &= ye^{xy}.2(-e^{-t}) + xe^{xy}.e^t \\ \text{or} &= -2ye^{xy-t} + xe^{xy+t} \\ &= -2e^t.e^{2-t} + 2e^2 \\ &= 0 \qquad\qquad (\text{i.e.} \frac{d}{dt}e^{xy} = \frac{d}{dt}e^2 = 0).\end{aligned}$$

Solution 5.3. (a) We have that $f(x, y) = x^3 + 2xy, \quad x = t; \quad y = 1 + t^2$ and thus

$$f = t^3 + 2t(1 + t^2) = 2t + 3t^3$$

Therefore

$$\frac{df}{dt} = 2 + 9t^2.$$

(b)

$$\begin{aligned}\frac{df}{dt} &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \\ &= (3x^2 + 2y).1 + 2x(2t) \\ &= (3t^2 + 2(1 + t^2)) + 2t(2t) \\ &= 2 + 9t^2.\end{aligned}$$

Solution 5.4. (a) $z - 16x - 2y = -95; \qquad \frac{2-x}{16} = \frac{1-y}{2} = z + 61$

(b) $z = x$

(c) $z = 0; \qquad x = y = 0,\ z$ arbitrary

(d) $40x + 16y - z - 36 = 0; \qquad \frac{x-5}{40} = \frac{y+8}{16} = 36 - z$

(e) $x + \sqrt{3}y + z - 1 - \frac{1}{\sqrt{3}}\pi = 0; \qquad x = \frac{3y-\pi}{3\sqrt{3}} = z - 1.$

Solution 5.5. (a) $\nabla f(0, 1) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

(b) $\nabla f(\pi, \pi/4) = \begin{bmatrix} 1/\sqrt{2} \\ 0 \end{bmatrix}$

(c) $\nabla f(2, 1) = \begin{bmatrix} 11 \\ 0 \end{bmatrix}.$

Solution 5.6. (a) $\frac{1}{2\sqrt{6}}$

(b) 1

(c) -1

(d) $2(3+\sqrt{3})$

(e) $\frac{\sqrt{3}}{10}(3\pi - \frac{2}{3})$

(f) $-\frac{80}{\sqrt{13}}e^6$.

Solution 5.7. (a) Most rapid increase: $\frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$. Most rapid decrease: $-\frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

(b) $\frac{1}{\sqrt{10}}(3,-1)$

(c) $\frac{1}{\sqrt{2}}(1,1)$ and $-\frac{1}{\sqrt{2}}(1,1)$

(d) (i) Zero in the direction of $\frac{1}{\sqrt{17}}(4,-1)$ and $\frac{1}{\sqrt{17}}(-4,1)$.

(ii) Greatest rate of change in the direction of $\frac{1}{\sqrt{17}}(1,4)$ and $\frac{1}{\sqrt{17}}(-1,-4)$.

The directions of (i) are orthogonal to those of (ii).

Solution 5.8. (a) $6/5$; (b) $-5/6$.

Solution 5.9. The normal vector to the hyperboloid is $\begin{bmatrix} 2x, -4y, -8z \end{bmatrix}$ and we want this equal to $\pm k(4,-2,4)$ for some real k, so we want

$$2x = \pm 4k, -4y = \pm - 2k, -8z = \pm 4k; \quad x = \pm 2k, y = \pm\frac{1}{2}k, z = \mp\frac{1}{2}k.$$

To find k we substitute into the equation of the hyperboloid, so that

$$4k^2 - \frac{1}{2}k^2 - k^2 = 16, \quad k^2 = \frac{32}{5}, k = \frac{4\sqrt{2}}{\sqrt{5}}$$

and so we have two points

$$(2k, \frac{1}{2}k, -\frac{1}{2}k), \quad (-2k, -\frac{1}{2}k, \frac{1}{2}k)$$

that is

$$\sqrt{\frac{8}{5}}(4,1,-1), \qquad \sqrt{\frac{8}{5}}(-4,-1,1).$$

Solution 5.10. (a) $\begin{bmatrix} 6x & 3 \\ 3 & -6y \end{bmatrix}$

(b) $\begin{bmatrix} 12x^2 & 0 \\ 0 & 6y \end{bmatrix}$

(c) $\begin{bmatrix} -\cos x & 0 \\ 0 & -\cos y \end{bmatrix}$

(d) $\begin{bmatrix} 2 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution 5.11. (a) For a stationary point $f_x = 0$ and $f_y = 0$. Thus for $f(x, y) = -4x^2 + xy - y^2$ we have

$$\begin{aligned} -8x + y &= 0, \\ x - 2y &= 0. \end{aligned}$$

which have one solution $(0, 0)$. To check the nature of the stationary point we need f_{xx}, f_{yy} and f_{xy}:

$$\begin{aligned} f_{xx} &= -8 \\ f_{yy} &= -2 \\ f_{xy} &= 1 \end{aligned}$$

As $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ the second derivative test gives $(0, 0)$ as a local maximum.

(b) Similarly to (a):

$$\begin{aligned} g_x &= \exp(x+y)\left(\frac{-3}{x^4y^2} + \frac{1}{x^3y^2}\right) = 0 \\ g_y &= \exp(x+y)\left(\frac{-2}{x^3y^3} + \frac{1}{x^3y^2}\right) = 0 \end{aligned}$$

which give

$$\begin{aligned} -3 + x &= 0 \\ -2 + y &= 0 \end{aligned}$$

and thus $(3, 2)$ is a stationary point. For the second derivative test we need

$$\begin{aligned} g_{xx} &= \exp(x+y)\left(\frac{12}{x^5y^2} - \frac{6}{x^4y^2} + \frac{1}{x^3y^2}\right) = e^5/324 \\ g_{yy} &= \exp(x+y)\left(\frac{6}{x^3y^4} - \frac{4}{x^3y^3} + \frac{1}{x^3y^2}\right) = e^5/216 \\ g_{xy} &= \exp(x+y)\left(\frac{6}{x^4y^3} - \frac{2}{x^3y^3} - \frac{3}{x^4y^2} + \frac{1}{x^3y^2}\right) = 0. \end{aligned}$$

Thus $g_{xx} > 0$ and $g_{xx}g_{yy} - g_{xy}^2 > 0$ and hence $(3, 2)$ is a minimum.

Solution 5.12. (a) $f(x, y) = x^3 + 3xy - y^3$ has first derivatives

$$f_x = 3x^2 + 3y, \quad f_y = 3x - 3y^2$$

and the Hessian matrix is

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & 3 \\ 3 & -6y \end{bmatrix}$$

The first derivatives are zero when

$$x^2 = -y, y^2 = x; \quad y^4 = x^2 = -y, y^4 + y = 0, y(y^3 + 1) = 0; y = 0, -1$$

and then we have for $y = 0$ the value $x = y^2 = 0$; and for $y = -1$ the value $x = y^2 = 1$. Hence there are two points

$$(0, 0), \det H = \det \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} = -9$$

$$(1, -1), \det H = \det \begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix} = 27$$

so that $(0, 0)$ is a saddle point and $(1, -1)$ is a local minimum as $f_{xx} = 6 > 0$.

(b) $f(x, y) = \cos x + \cos y$ has first partial derivatives

$$f_x = -\sin x, \quad f_y = -\sin y$$

and the Hessian is

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} -\cos x & 0 \\ 0 & -\cos y \end{bmatrix}.$$

The first derivatives are zero when

$$\sin x = 0, \sin y = 0; (x, y) = (n\pi, m\pi)$$

for integers n, m. At the stationary point $(n\pi, m\pi)$, the Hessian is

$$H = \begin{bmatrix} -(-1)^n & 0 \\ 0 & -(-1)^m \end{bmatrix}$$

with determinant $\det H = (-1)^{n+m}$.

If $n + m$ is odd then $(n\pi, m\pi)$ is a saddle point.

If n and m are both even then $(n\pi, m\pi)$ is a local maximum.

If n and m are both odd then $(n\pi, m\pi)$ is a local minimum.

Solution 5.13. The profit $P(x,y)$ is $P(x,y) = R(x,y) - C(x,y)$ whose stationary points are required. Thus

$$P_x = 2 - (2x - 2y + 6) = 0$$
$$P_y = 3 - (-2x + 4y - 9) = 0$$

which have solution $(2,4)$.

Check type: $P_{xx} = -2$, $P_{yy} = -4$ and $P_{xy} = 2$ thus at $(2,4)$ $P_{xx} < 0$ and $P_{xx}P_{yy} - P_{xy}^2 > 0$ and hence $(2,4)$ is a maximum. The profit is \$15 million.

Solution 5.14. (a) $x^3 + x^2y + xy^5 + g(y)$ where g is an arbitrary function.

(b) $3x^2y + xy^2 + \frac{1}{6}y^6 + h(x)$ where h is an arbitrary function.

Solution 5.15. (a) $\frac{1}{2} + \pi$

(b) $\frac{1}{2}(e^2 - e)$

(c) $e - 2$

(d) $\frac{1}{9}\sin 1$.

Solution 5.16. $\frac{11}{2}$.

Solution 5.17. $1 - \frac{2}{\pi}$.

Solution 5.18. $(\bar{x}, \bar{y}) = \left(\frac{3}{4}, 2\right)$. [Note $m = 1.5$.]